Angular Momenta

In this section, we begin the study of the quantum theory of angular momentum, concentrating initially on orbital angular momentum. The approach taken here is algebraic, i.e. we try to derive as many things as possible from the algebra of the angular momentum operators. Suppose our system is one particle in three dimensions. The orbital angular momentum operator is

$$\vec{L} = \vec{X} \times \vec{P},$$

or

$$L_1 = X_2 P_3 - X_3 P_2, \text{ etc}$$

Here X_j and P_k are the usual coordinate and momentum operators, satisfying

$$[P_j, X_k] = -i\hbar\delta_{jk}$$

From this equation, we obtain

$$[L_j, L_k] = i\hbar\epsilon_{jkn} L_n,$$

or

$$[L_1, L_2] = i\hbar L_3, \text{ etc}$$

In what follows, we will not refer to configuration space very often. However, we do need one result, namely the formula for L_3 in spherical coordinates. The wave function in spherical coordinates is $< r, \theta, \phi|\Psi >$. The action of L_3 is specified by

$$< r, \theta, \phi|L_3|\Psi >= \frac{\hbar}{i}\frac{\partial}{\partial\phi} < r, \theta, \phi|L_3|\Psi >$$

If the particle is in an eigenstate of L_3 the ϕ dependence of the wave function will be

$$< r, \theta, \phi|L_3|\Psi > \sim \exp(im\phi),$$

where $\hbar m$ is the eigenvalue of L_3. For the motion of a particle, we demand that the wave function be periodic in ϕ, so we must have

$$\exp(im2\pi) = 1,$$

or m must be an integer (positive or negative or zero.) So when the type of angular momentum under discussion is orbital, the third component of angular momentum has eigenvalues which are integral multiples of $\hbar$.

Scalar and Vector Operators

It is familiar in classical physics, that if we perform a rigid rotation of a system, that certain quantities remain unchanged whereas others do change in a regular way. Quantities which remain unchanged under a rotation are called (rotational) *scalars*. An example is the kinetic energy of a particle. If we rotate the "system" by rotating the velocity vector, the kinetic energy does not change. On the other hand, the components of the velocity certainly do change. (**NOTE** In all of our discussion of rotations, we will take the so-called "active" viewpoint, where the rotation is applied to the object of interest, and the coordinate system is held fixed.) If we use a $\prime$ symbol to denote the rotated object, and no $\prime$ to denote the original, then for a scalar quantity, we have

$$S' = S. \tag{1}$$

For a vector quantity, let us consider how the components transform if the system is rotated (counter-clockwise) around the 3-axis by an angle α. By drawing a few pictures, it is easy to see that the correct transformation properties for the components of a vector $\vec{V}$ are given by

$$\begin{aligned} V_1' &= \cos\alpha V_1 - \sin\alpha V_2 \\ V_2' &= \cos\alpha V_2 + \sin\alpha V_1 \\ V_3' &= V_3 \end{aligned} \tag{2}$$

Now let us turn to quantum mechanics and require that expected values of scalar and vector operators transform as in Eqs.(1) and (2). In the active viewpoint, we consider that transformed state $|\Psi' >$, which is related to the original state by the unitary transformation representing (in the present case) the rotation. For simplicity, we will continue to take our rotation to be one of angle α around the 3-axis. So for a scalar, we have

$$< S >' \equiv < \Psi'|S|\Psi' > = < \Psi|S|\Psi > \equiv < S > .$$

From transformation theory, the relation between $|\Psi' >$ and $|\Psi >$ is

$$|\Psi' > = \exp(-\frac{iL_3\alpha}{\hbar})|\Psi >$$

Using our equation, we have

$$< \Psi'|S|\Psi' > = < \Psi| \exp(\frac{iL_3\alpha}{\hbar}) S \exp(-\frac{iL_3\alpha}{\hbar})|\Psi > = < \Psi|S|\Psi >,$$

or stripping off the states,

$$\exp(\frac{iL_3\alpha}{\hbar}) S \exp(-\frac{iL_3\alpha}{\hbar}) = S.$$

Now, if we differentiate, we get

$$i\hbar\partial_\alpha \exp(\frac{iL_3\alpha}{\hbar}) S \exp(-\frac{iL_3\alpha}{\hbar}) = \exp(\frac{iL_3\alpha}{\hbar})[S, L_3] \exp(-\frac{iL_3\alpha}{\hbar}) = 0,$$

since the right hand side of the equation is independent of α. The result is

$$[S, L_3] = 0.$$

Using other axes of rotation, we will clearly get

$$[S, L_k] = 0, \ k = 1, 2, 3.$$

The conclusion is that an operator representing a rotational scalar must commute with all components of the angular momentum.

For a vector operator, let us take the components of $\vec{X}$. Again taking the rotation to be one of angle α around the 3 axis, from Eqs.(2), we have

$$\begin{aligned} < \Psi| \exp(\frac{iL_3\alpha}{\hbar}) X_1 \exp(-\frac{iL_3\alpha}{\hbar}) |\Psi > &= \cos\alpha < \Psi|X_1|\Psi > - \sin\alpha < \Psi|X_2|\Psi > \quad (3) \\ < \Psi| \exp(\frac{iL_3\alpha}{\hbar}) X_2 \exp(-\frac{iL_3\alpha}{\hbar}) |\Psi > &= \cos\alpha < \Psi|X_2|\Psi > + \sin\alpha < \Psi|X_1|\Psi > \\ < \Psi| \exp(\frac{iL_3\alpha}{\hbar}) X_3 \exp(-\frac{iL_3\alpha}{\hbar}) |\Psi > &= < \Psi|X_3|\Psi > \end{aligned}$$

As in the case of the scalar operator, these equations must hold for an arbitrary state. This is guaranteed if we demand that they hold for the operators themselves. Writing out the first of Eqs.(3), we have

$$\exp(\frac{iL_3\alpha}{\hbar}) X_1 \exp(-\frac{iL_3\alpha}{\hbar}) = \cos\alpha X_1 - \sin\alpha X_2$$

Differentiating both sides with respect to α, we obtain

$$\exp(\frac{iL_3\alpha}{\hbar}) [X_1, L_3] \exp(-\frac{iL_3\alpha}{\hbar}) = -i\hbar(\sin\alpha X_1 + \cos\alpha X_2).$$

This equation holds for all α. If we set $\alpha = 0$ we obtain

$$[L_3, X_1] = i\hbar X_2.$$

Using other axes of rotation and considering an arbitrary three-vector operator $\vec{V}$, we can derive

$$[L_j, V_k] = i\hbar\epsilon_{jkn} V_n. \quad (4)$$

So turning the argument around, we can *define* a three-vector operator as one whose components satisfy Eq.(4). If we have a single particle, then the list of vector operators is $\vec{X}, \vec{P}, \vec{L}$.

To summarize, demanding that quantities transform under rotations in the same way as they do in classical physics leads to a definite form for the commutation relations between the operators representing these quantities and the angular momentum of the system. The discussion was carried out for a single particle, but applies to a arbitrarily complicated quantum system.

Eigenvalues of L^2

We will determine the eigenvalues of L^2 by making use of the operators $L_\pm$, defined by

$$L_\pm = L_1 \pm L_2.$$

The standard angular momentum commutation rules give

$$[L_3, L_\pm] = \pm\hbar L_\pm$$

The action of $L_\pm$ changes the eigenvalue of L_3 by $\pm\hbar$. For example,

$$L_3(L_+|m>) = (L_+L_3 + [L_3, L_+])|m> = \hbar(mL_+|m> + L_+|m>) = \hbar(m+1)(L_+|m>).$$

We will get our result for the eigenvalue of L^2 by expressing L^2 in terms of the $L_\pm$. We have

$$L_+L_- = (L_1 + iL_2)(L_1 - iL_2) = L_1L_1 + L_2L_2 - i[L_1, L_2] = L_1L_1 + L_2L_2 + \hbar L_3, \quad (5)$$

and

$$L_-L_+ = L_1L_1 + L_2L_2 - \hbar L_3$$

Relating these quantities to L^2, we have

$$L_+L_- = L^2 - L_3L_3 + \hbar L_3, \quad (6)$$

and

$$L_-L_+ = L^2 - L_3L_3 - \hbar L_3. \quad (7)$$

Now assume the eigenvalue of L^2 is a real number $\lambda \geq 0$, and consider the matrix element $< \lambda, m|L_+L_-|\lambda, m >$. Inserting a complete set of eigenstates of L_3, we have

$$< \lambda, m|L_+L_-|\lambda, m > = \sum_{m'} < \lambda, m|L_+|\lambda, m' >< \lambda, m'|L_-|\lambda, m > = \sum_{m'} | < \lambda, m'|L_-|\lambda, m > |^2$$

From the last equality, we have that $< \lambda, m|L_+L_-|\lambda, m > \geq 0$. Using Eq.(5), we have

$$< \lambda, m|L_+L_-|\lambda, m > = < \lambda, m|L^2 - L_3L_3 + \hbar L_3|\lambda, m > = \lambda + \hbar^2(m - m^2).$$

It is clear that this expression will go negative for large enough m^2. The resolution is that there must be a minimum value of m, denoted as MIN, for which $L_-|\lambda, MIN >= 0$. We then have

$$0 = \lambda + \hbar^2(MIN - MIN^2), \text{ or } \lambda = \hbar^2 MIN(MIN - 1)$$

A parallel argument applied to the matrix element of $< \lambda, m|L_-L_+|\lambda, m >$ shows that there must be a maximum value of m, denoted as MAX, and related to λ by

$$\lambda = \hbar^2 MAX(MAX + 1).$$

The two expressions we have for λ must agree. The only acceptable relation between MAX and MIN is $MIN = -MAX$. It is normal to set $MAX = l$, so the value of L^2 becomes

$$L^2 = \hbar^2 l(l+1), \; l = 0, 1, 2, \ldots, \quad (8)$$

and states are labeled as $|l, m>$. The allowed values of m are then

$$-l \leq m \leq l. \quad (9)$$

Matrix Elements of $L_{\pm}$.

From Eq.(7), we can write

$$< l,m|L_-L_+|lm >= | < l,m+1|L_+|l,m > |^2 = hbar^2[l(l+1)-m(m+1)] = \hbar^2[(l-m)(l+m+1)].$$

Choosing a phase and taking the square root, we have

$$< l,m+1|L_+|l,m >= \hbar\sqrt{(l-m)(l+m+1)}. \tag{10}$$

Finally, replacing m by $m-1$, we obtain

$$< l,m|L_+|l,m-1 >=< l,m-1|L_-|l,m >= \hbar\sqrt{(l+m)(l-m+1)} \tag{11}$$

At this point, we have the full set of results. The eigenvalues of L^2 are given in Eq.(8). Eq.(9) allows L_3 to be written out as a $(2l+1)\times(2l+1)$ matrix, and finally by taking combinations of Eqs.(6) and (7) the matrices for L_1 and L_2 can be constructed. With the (standard) choice of phases we have made, the matrix for L_1 is purely real, while that for L_2 is purely imaginary.

It is useful to work out a simple example. Consider the case of $l=1$. The matrices are 3×3 for this case. The matrix for L_3, is

$$L_3 = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

From Eq.(10), we have

$$L_+|1,0 >= \hbar\sqrt{2}|1,1 >, \quad L_+|1,-1 >= \hbar\sqrt{2}|1,0 > .$$

These results allows us to obtain

$$L_+ = \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

Taking the adjoint, we have

$$L_- = \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

From these last two equations, L_1 and L_2 are obtained as

$$L_1 = \frac{1}{2}(L_+ + L_-), \quad L_2 = \frac{1}{2i}(L_+ - L_-)$$

Beyond Orbital Angular Momentum

In the above sections, the fact that the angular momentum being discussed was *orbital* came in when we restricted the value of the third component of angular momentum to be an integer multiple of $\hbar$. This gives integer values of l and angular momentum matrices which are $(2l+1) \times (2l+1)$ dimensional. This does not exhaust the possible sets of matrices satisfying angular momentum commutation rules. To see this suppose we have an angular momentum vector $\vec{J}$, with the following commutation rules,

$$[J^2, J_k] = 0, \;\; [J_k, J_n] = i\hbar\epsilon_{knp}J_p.$$

Without assuming J_3 has eigenvalues which are integral multiples of $\hbar$, all the steps of previous sections go through. In particular there must be a maximum value of J_3, denoted as MAX, and a minimum value, denoted as MIN, where $MIN = -MAX$, and setting $j = MAX$, we have that the eigenvalue of J^2 can be written as

$$\hbar^2 j(j+1).$$

The final question is, what are the allowed values of j? To settle this question, imagine starting at the state $|j, MIN>$ and successively applying the raising operator J_+ until the state $|j, MAX>$ is reached. This must take an integer number of steps. Call the integer n. Then we can write

$$MAX = MIN + n.$$

Using $MIN = -MAX$, we have

$$MAX = \frac{n}{2}.$$

This allows half-integer solutions, so choosing $j = 1/2, 3/2, 5, 2, \ldots$ leads to a set of even dimensional matrices which satisfy all the angular momentum commutation rules. It is easy to check that trying to find further fractions, e.g. $MAX = 1/4$, cannot work.

Summary Including all possible cases, the total angular momentum quantum number is restricted to $j = 0, 1/2, 1, 3/2, 2, \ldots$. The square of the angular momentum takes values

$$\hbar^2 j(j+1),$$

while J_3 has eigenvalues $\hbar m$, with

$$-j \leq m \leq j.$$

The matrix elements of $J_\pm$ satisfy

$$< j, m+1|J_+|j, m >= \hbar\sqrt{(j-m)(j+m+1)}, \tag{12}$$

and

$$< j, m-1|J_-|j, m >= \hbar\sqrt{(j+m)(j-m+1)} \tag{13}$$

The simplest example is $j = 1/2$, where we have

$$J_3 = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Using Eqs.(12) and (13), we have

$$J_+ = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and

$$J_- = \frac{\hbar}{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

so

$$J_1 = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$J_2 = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The coefficients of $\hbar/2$ in the formulas for J_1, J_2, J_3, are the famous Pauli matrices $\sigma_1, \sigma_2, \sigma_3$.