1 Classical Mechanics

Classical mechanics cannot explain quantum phenomena like the stability of atoms, sharp atomic spectral lines, the spectral distribution of black body radiation, chemical reactions, etc. Nevertheless, classical and quantum mechanics are closely connected. Just as geometrical optics can be regarded as the short wavelength limit of wave optics, so classical mechanics can be regarded as the short wavelength limit of quantum mechanics. The formulation of quantum mechanics by Hamilton is the most useful for seeing the connections between classical and quantum mechanics. Schrödinger made brilliant use of Hamilton's ideas in writing down his wave equation. Dirac and Feynman used Hamiton's princple to formulate the path integral approach to quantum mechanics.

Lagrangians We are used to thinking of the Lagrangian of a system as an elegant way to find the equations of motion. The system in general may have several coordinates, usually denoted as q_k. The q_k are usually linear or angular coordinates, but they may be more general. The Lagrangian depends on the q_k, and their time derivatives, usually denoted as $\dot{q}_k$, i.e.

$$\dot{q}_i \equiv \frac{dq_i}{dt}.$$

From the generalized coordinates and the Lagrangian, we define generalized momenta by

$$p_k = \frac{\partial L}{\partial \dot{q}_k}. \tag{1}$$

In terms of the q_k and p_k, the equations of motion are

$$\dot{p}_k = \frac{\partial L}{\partial q_k}. \tag{2}$$

Here are some sample Lagrangians

- Harmonic oscillator

$$L = \frac{m}{2}\left(\dot{q}^2 - \omega^2 q^2\right)$$

- Forced harmonic oscillator

$$L = \frac{m}{2}\left(\dot{q}^2 - \omega^2 q^2\right) + F(t)q$$

- Particle constrained to a sphere of radius a.

$$L = \frac{m}{2}a^2\left(\dot{\theta}^2 + (\sin\theta)^2\dot{\phi}^2\right)$$

- Central force motion in a plane

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)$$

- Central force motion in three dimensions

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - V(r)$$

Note that there is nothing wrong with having an explicitly time dependent force or potential in the Lagrangian. The harmonic oscillator with a time-dependent force is an interesting model for more complicated systems.

Hamiltonians Given a Lagrangian, one can define a Hamiltonian,

$$H(q_k, p_k) = \sum_k p_k\dot{q}_k - L, \tag{3}$$

where as the notation implies, in H we eliminate the $\dot{q}_k$ in favor of the p_k. Using the Hamiltonian, the equations of motion are

$$\begin{aligned}\dot{q}_k &= \frac{\partial H}{\partial p_k} \\ \dot{p}_k &= -\frac{\partial H}{\partial q_k}\end{aligned} \tag{4}$$

Exercise 1 Find the Hamiltonians for the Lagrangians listed above. Write down the equations of motion in Hamiltonian form.

Poisson Brackets Quantum mechanics is full of commutators and commutation relations. It is less familiar that this type of structure also exists in classical mechanics, through a quantity known as the Poisson bracket. Suppose we are in the Hamiltonian formulation of classical mechanics, and we have two dynamical quantities A and B, both of which are functions of coordinates and momenta. The Poisson bracket of A with B is defined to be

$$\{A, B\} \equiv \sum_{p_k, q_k}\left(\frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q_k}\right) \tag{5}$$

From the definition, it follows that the Poisson bracket of two quantities is anti-symmetric,

$$\{A, B\} = -\{B, A\}. \tag{6}$$

Poisson brackets also satisfy many of the same rules that apply to ordinary derivatives. For example,

$$\{AB, C\} = A\{B, C\} + B\{A, C\} \tag{7}$$

Poisson brackets for any three quantities also satisfy the so-called Jacobi identity,

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0 \tag{8}$$

The time rate of change of any quantity O is given by its Poisson bracket the Hamiltonian,

$$\frac{dO}{dt} = \{O, H\}. \tag{9}$$

For example it is easy to show using the definition of Poisson brackets, that

$$\dot{q}_k = \{q_k, H\} = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = \{p_k, H\} = -\frac{\partial H}{\partial q_k} \tag{10}$$

Exercise 2 Consider an arbitrary function of the coordinates and momenta, O. Assume O does not have explicit time dependence. Derive the equation of motion for O (dO/dt) by differentiating O and using Hamilton's equations, Eq.(4). From this result and the definition of the Poisson bracket, show that O satisifies Eq.(9).

Let us compute the Poisson brackets for a few familiar quantities. We have

$$\{q_k, p_j\} = \delta_{kj}. \tag{11}$$

Suppose we have a particle moving in three dimensions. It is natural to use the cartesian quantities x_k and p_j to describe coordinates and momenta. The orbital angular momentum is $\vec{L}$, given by

$$\vec{L} = \vec{x} \times \vec{p}, \tag{12}$$

or

$$L_1 = x_2 p_3 - x_3 p_2, \quad L_2 = x_3 p_1 - x_1 p_3, \quad L_3 = x_1 p_2 - x_2 p_1$$

For the Poisson brackets of different components of $\vec{L}$ it is easy to show that

$$\{L_j, L_k\} = \epsilon_{jkl} L_l \tag{13}$$

Exercise 3 Check Eq.(13) for the cases $j = 1, k = 2$ and $j = 3, k = 2$.

These examples show that the algebraic structure of quantum mechanics has a counterpart in classical mechanics. In quantum mechanics, dynamical quantities like the components of orbital angular momentum become operators. Dirac proposed that in the transition from classical mechanics to quantum mechanics, the Poisson bracket should be replaced by a multiple of the quantum commutator. Specifically, Dirac's proposal was

$$\{A, B\} \longrightarrow \frac{-i}{\hbar}[\mathbf{A}, \mathbf{B}]. \tag{14}$$

In Eq.(14), the curly brackets $\{\cdot, \cdot\}$ imply a Poisson bracket, while the ordinary brackets $[\cdot, \cdot]$ imply a quantum commutator, and we have denoted the operators corresponding to the classical quantities A and B by $\mathbf{A}$ and $\mathbf{B}$.

Dirac's proposal is accurate for fundamental relations like Eq.(13). It is also generally safe to use it in Cartesian coordinates. However, the replacement of the Poisson bracket between a coordinate and a momentum, Eq.(11) by $-i/\hbar$ times the quantum commutator can cause problems in curvilinear coordinates. This will be explored in a later section of these notes.

Action We are used to the Hamiltonian playing the starring role in quantum mechanics. However, in many modern applications, path integrals, quantum field theory, etc. it is the Lagrangian that is most important, and in particular the *action*, the properties of which are explored in the next few sections. The action, denoted as S, is defined as

$$S = \int_1^2 L(q_k, \dot{q}_k) dt, \tag{15}$$

i.e. the time integral of the Lagrangian between fixed endpoints. Fixed endpoints means that the initial ($q_k(t_1)$) and final ($q_k(t_2)$) values of the coordinates held fixed. To visualize what this means, think of a pitcher in baseball, throwing a baseball intended to hit a specific location in the strike zone. The initial coordinate is the location of the ball as it leaves the pitcher's throwing hand. The final coordinate is the location of the ball when it reaches the target in the strike zone. The time interval would be the time it takes for the pitch to travel on its path, about 0.5 seconds for a very good fastball. As opposed to thinking about initial positions and intial velocities, in considering the action one thinks of inital and final coordinates.

The *classical path* is the solution of the equations of motion which connects the q_k from their initial to their final values. Hamilton's principle is the statement that the action is *stationary* to small variations around the classical path. To see what this means let us replace all the $q_k(t)$ by $q_k(t) + \delta q_k(t)$, where the δq_k are regarded as small and we will only keep quantities to first order in the δq_k. The requirement that the endpoints are fixed is imposed by demanding that $\delta q_k(t_1) = \delta q_k(t_2) = 0$. Let us write the variation of the action, keeping only first order terms. We have

$$\delta S = \int_1^2 \sum_k \left(\frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k + \frac{\partial L}{\partial q_k} \delta q_k \right) dt. \tag{16}$$

Now integrate the first term by parts, obtaining

$$\int_1^2 \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k = \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta q_k]_{t_1}^{t_2} - \int_1^2 \sum_k \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k. \tag{17}$$

But the first term vanishes because the δq_k vanish at $t = t_1$ and $t = t_2$. Going back to δS, we now have

$$\delta S = \int_1^2 \sum_k \left(-\frac{d}{dt} (\frac{\partial L}{\partial \dot{q}_k}) + \frac{\partial L}{\partial q_k} \right) \delta q_k dt. \tag{18}$$

Now the δq_k are infinitesimal but arbitrary. The only way to insure that δS vanishes is to demand that the coefficient of each δq_k vanish in the integrand, or

$$-\frac{d}{dt} (\frac{\partial L}{\partial \dot{q}_k}) + \frac{\partial L}{\partial q_k} = 0, \tag{19}$$

which are just the equations of motion. So if we consider the action as a function(al) of the $q_k(t)$, the statement is that the action becomes *stationary* to first order when the $q_k(t)$ obey the equations of motion, i.e. the $q_k(t)$ are on the *classical path.*

Free Particle and General Features of the Action To illustrate some features of the action, consider the motion of a free particle in one dimension, starting at x_1, arriving at x_2, over a time interval t. For a free particle, from elementary physics we have

$$x(t') = x_1 + vt', \tag{20}$$

Now the particle arrives at x_2 at $t' = t$, so

$$x_2 = x_1 + vt \tag{21}$$

which gives

$$v = \frac{x_2 - x_1}{t}. \tag{22}$$

The Lagrangian is

$$L = \frac{m}{2}\dot{x}^2 = \frac{m}{2}v^2 \tag{23}$$

where v is of course constant. The action is then

$$S = \int_0^t L(x(t')dt' = \frac{m}{2}v^2 t = \frac{m}{2}\frac{(x_2 - x_1)^2}{t} \tag{24}$$

Exercise 4 Consider the action for the varied path

$$x(t') = x_1 + vt' + \delta x(t')$$

where

$$\delta x(t') = \epsilon \sin(\frac{\pi t'}{t}).$$

Compute the action for the varied path. Show that the term of $O(\epsilon)$ vanishes. Is the action a min or a max here?

Although this example is very simple, the action of Eq.(24) allows illustration of several general properties of the action. The action has been computed with fixed endpoints, over a fixed time interval. However, important physical quantities are delivered by differentiating with respect to these quantities. Let us first take the derivative with respect to the final endpoint, x_2. We have

$$\frac{\partial S(x_2, x_1, t)}{\partial x_2} = m\frac{(x_2 - x_1)}{t} = p_2, \tag{25}$$

where p_2 is the final momentum. Likewise, if we differentiate with respect to x_1, we get

$$\frac{\partial S(x_2, x_1, t)}{\partial x_1} = -m\frac{(x_2 - x_1)}{t} = -p_1, \tag{26}$$

where p_1 is the initial momentum (of course the same as the final here.) Both of these results generalize, and are rather easy to establish. To do so, return to Eq.(17), and now allow the δq_k to be non-vanishing at the end points. If the $q_k(t)$ satisfy the equations of motion, the variation of the classical action will only come from the endpoints, and we have

$$\delta S = \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta q_k]_{t_1}^{t_2} = \sum_k \left(p_k(t_2)\delta q_k(t_2) - p_k(t_1)\delta q_k(t_1) \right), \tag{27}$$

where we used Eq.(1). Eq.(27) is equivalent to the equations

$$\begin{aligned} \frac{\partial S(q_k(2), q_k(1), t)}{\partial q_k(2)} &= p_k(2) \\ \frac{\partial S(q_k(2), q_k(1), t)}{\partial q_k(1)} &= -p_k(1) \end{aligned} \tag{28}$$

It is important to emphasize that the δS referred to in Eqs.(28) is the variation of the action computed for the classical path, around its endpoints. If we were not at the classical path, there would still be an integral over time.

The other derivative we can take is with respect to time. For our free particle, we get

$$\frac{\partial S}{\partial t} = -\frac{m}{2} \frac{(x_2 - x_1)^2}{t^2}, \tag{29}$$

which is minus the energy of the particle. This is again easy to establish in general. We are going to vary the action computed for the classical path by changing the time interval slightly. We will vary only the final time. $t \longrightarrow t + dt$. So imagine the classical path that hits the same final point, but at $t + dt$. The extra bit in the integral will contribute a term

$$L(2)dt$$

to δS. But we also need to include the term that comes when we integrate by parts,

$$\sum_k p_k(2)\delta q_k(2).$$

Now the $\delta q_k(2)$ are not arbitrary, they must be chosen so the particle arrives at $q_k(2)$ at time $t + dt$.

Exercise 5 Show that the requirement that the particle arrives at $q_k(2)$ at time $t + dt$ is satisfied by

$$\delta q_k(2) + \dot{q}_k(2)dt = 0$$

Putting the terms in δS together, we have

$$\delta S = -\left(\sum_k p_k(2)\dot{q}_k(2) - L \right) dt, \tag{30}$$

which establishes that

$$-\frac{\partial S(q_k(2), q_k(1), t)}{\partial t} = H, \tag{31}$$

where H is the Hamiltonian of the system, which equals the energy if the potential is time-independent.

Hamilton-Jacobi Equation Using the results of the previous section we can write an important differential equation involving the action. This is an equation satisfied by the *classical* action, meaning the action for starting at one point, and ending at another, with the system connecting the two points being on its *classical path*. We write the result for a particle of mass m moving in three dimensions ($k = 1, 2, 3,$) in a potential V. We have

$$\frac{\partial S}{\partial t} + \sum_k \frac{1}{2m}\left(\frac{\partial S}{\partial q_k}\right)^2 + V(q_k) = 0. \tag{32}$$