Creation and Destruction Operators and Coherent States

WKB Method for Ground State Wave Function   We first rewrite the ground state harmonic oscillator wave function,

$$< x|0> = (\frac{m\omega}{\pi\hbar})^{1/4} \exp(-\frac{m\omega x_a^2}{2\hbar}) \tag{1}$$

In the notes on imaginary time path integrals, we obtained this formula from the imaginary time propagator for the harmonic oscillator. Before introducing creation and destruction operators, let us explore the exponential factor in the ground state wave functions. The method used is called the WKB (for Wentzel, Kramers, and Brillouin) method and involves a quantity closely related to the action used in the path integral discussion. We start by assuming the following form for a bound state wave function,

$$\Psi = \exp(-\frac{W(x)}{\hbar}). \tag{2}$$

Note that W has the same dimensions as $\hbar$ and the action S. Then we have

$$\partial_x \Psi = -\frac{\partial_x W}{\hbar}\Psi,$$

and

$$\partial_x^2 \Psi = \left[(\frac{\partial_x W}{\hbar})^2 - \frac{\partial_x^2 W}{\hbar} \right] \Psi$$

Let us use this formula in the Schrödinger equation, allowing an arbitrary potential for the moment. From

$$\left(-\frac{\hbar^2}{2m}\partial_x^2 + V(x) \right) \Psi = E\Psi,$$

we obtain

$$\left(-\frac{(\partial_x W)^2}{2m} + V(x) + \hbar\frac{\partial_x^2 W}{2m} \right) \Psi = E\Psi \tag{3}$$

Note that the first two terms on the left side are independent of $\hbar$. The method we are using is *semiclassical* in which case it makes sense to try to solve the equation order by order in $\hbar$. Now consider the eigenvalue E, in particular, suppose we are looking for the ground state of the system. For an oscillator the classical lowest energy is zero, and the actual ground state is $\hbar\omega/2$. Let us follow this suggestion and assume for other potentials as well that for the ground state $E \sim O(\hbar)$. Then if we make the terms independent of $\hbar$ cancel, we get

$$\frac{(\partial_x W)^2}{2m} = V(x),$$

or

$$\partial_x W = \sqrt{2mV(x)}$$

which implies that

$$W(x) = \int_{x_0}^{x} dx' \sqrt{2mV(x')}$$

The choice of x_0 is not completely specified. However, if we are looking for the ground state of the system, it makes sense to choose x_0 at the minimum of $V(x)$. Returning to the oscillator, we obtain

$$W(x) = \int_{0}^{x} dx' \sqrt{2mV(x')} = \int_{0}^{x} dx' m\omega x' = \frac{1}{2} m\omega x^2,$$

which when substituted in Eq.(2) gives the correct form of the oscillator ground state wave function. If we go further and match the $O(\hbar)$ terms in Eq.(3), assuming E is proportional to $\hbar$, we get

$$\hbar \frac{\partial_x^2 W}{2m} = E.$$

Using $W(x) = m\omega x$, this gives

$$E = \frac{\hbar \omega}{2},$$

which is the correct answer for the oscillator. This general method can be applied to other smooth potentials, but it is only exact for the oscillator. We will not pause to study the function $W(x)$ further. The relation of W to the action S is basically that one trades the time t for the energy E, by making a so-called Legendre transform. More on this can be found in *"Classical Mechanics"*, by H. Goldstein. There is also much more to the WKB method than we have discussed here. It is more typical to use it for highly excited states of a particle moving in a smooth potential. In that case the particle can be in both the classically allowed region and the classically forbidden region. For the classically allowed region the form

$$\Psi = \exp(i \frac{W}{\hbar})$$

is the appropriate one to use, and the factor of i in the exponent is replacedd by -1 when x moves into the classically forbidden region. For more on WKB and semiclassical quantum mechanics in general see Shankar, sec. 16.2, Gottfried and Yan, sec.2.8 and Chapter VII of *"Quantum Mechanics,"* by Landau and Lifshitz.

Creation and Destruction Operators   Now let us turn to the creation and destruction operators using our solutions of the Heisenberg equations of motion. Rewriting our previous results,

$$X(t) = X \cos \omega_0 t + \frac{P}{m\omega_0} \sin \omega_0 t, \tag{4}$$

and

$$P(t) = P \cos \omega_0 t + m\omega_0 X \sin \omega_0 t. \tag{5}$$

By taking matrix elements of these equations between eigenstates, we found by matching time dependence on both sides,

$$< n - 1|X(t)|n >= \exp(-i\omega t) < n - 1|(X + i\frac{P}{m\omega})|n >, \ n \geq 1$$

$$< n - 1|(X - i\frac{P}{m\omega})|n >= 0, \ all \ n,$$

and

$$< n + 1|X(t)|n >= exp(i\omega t) < n + 1|(X - i\frac{P}{m\omega})|n \geq 0,$$

$$< n - 1|(X + i\frac{P}{m\omega})|n >= 0, \ all \ n,$$

and further there is a set of states with energies $\hbar\omega(n + 1/2), \ n = 0, 1, 2, \ldots$. It is clear from these results that the operator

$$(X + i\frac{P}{m\omega})$$

can only lead to $|n - 1 >$ when acting on $|n >$, while

$$(X - i\frac{P}{m\omega})$$

can only lead to $|n+1 >$ when acting on $|n >$. Further since all states must have positive energy, we must have

$$(X + i\frac{P}{m\omega})|0 >= 0$$

Let us multiply these operators. We have

$$(X - i\frac{P}{m\omega})(X + i\frac{P}{m\omega}) = X^2 + \frac{P^2}{(m\omega)^2} - \frac{1}{m\omega}[PX - XP]$$

Multiplying by $m\omega^2/2$ and using $[PX - XP] = -i\hbar$, we have

$$\frac{1}{2}m\omega^2(X - i\frac{P}{m\omega})(X + i\frac{P}{m\omega}) = H - \frac{1}{2}\hbar\omega$$

Dividing by $\hbar\omega$, we have

$$\frac{H}{\hbar\omega} = \frac{1}{2} + \frac{m\omega}{2\hbar}(X - i\frac{P}{m\omega})(X + i\frac{P}{m\omega})$$

The right hand side is dimensionless. We now define

$$a \equiv \sqrt{\frac{m\omega}{2\hbar}}(X + i\frac{P}{m\omega}),$$

and

$$a^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}(X - i\frac{P}{m\omega}).$$

The commutator of a with $a^\dagger$ follows from the X, P commutator. We have

$$[a, a^\dagger] = \frac{m\omega}{2\hbar}[(X + i\frac{P}{m\omega}), (X - i\frac{P}{m\omega})] = \frac{i}{\hbar}[P, X] = 1.$$

The harmonic oscillator Hamiltonian in terms of a and $a^\dagger$ is

$$H = \hbar\omega(a^\dagger a + \frac{1}{2})$$

Rearranging Eqs.(4) and (5), we have

$$X(t) = \frac{1}{2}\left[(X - i\frac{P}{m\omega})\exp(i\omega t) + (X + i\frac{P}{m\omega})\exp(-i\omega t)\right],$$

or using the definitions of a and $a^\dagger$, we have

$$X(t) = \sqrt{\frac{\hbar}{2m\omega}}\left[a\exp(-i\omega t) + a^\dagger\exp(i\omega t)\right] \tag{6}$$

Writing this equation one more time,

$$X(t) = \sqrt{\frac{\hbar}{2m\omega}}\left[a(t) + a^\dagger(t)\right],$$

so we have

$$a(t) = a\exp(-i\omega t), \quad a^\dagger(t) = a^\dagger\exp(i\omega t),$$

where $a(t)$ and $a^\dagger(t)$ are the Heisenberg operators corresponding to the Schrödinger operators a and $a^\dagger$. The corresponding expressions for $P(t)$ follow from these,

$$P(t) = m\frac{dX}{dt} = -i\sqrt{\frac{\hbar\omega}{2m}}\left[a\exp(-i\omega t) - a^\dagger\exp(i\omega t)\right] \tag{7}$$

Expressions like Eqs.(6) and (7) occur in many more general problems, basically any problem that can be expanded in harmonic oscillators. Examples are photons, phonons, and the so-called "vector mesons" of particle physics, such as the W and Z particles.

Let us now find the matrix elements of a and $a^\dagger$. Using

$$E_n = \hbar\omega(n + \frac{1}{2}),$$

we have

$$< n|(H - \frac{\hbar\omega}{2})|n > = \hbar\omega n.$$

But
$$H - \hbar\omega\frac{1}{2} = \hbar\omega(a^\dagger a),$$
so we have
$$\hbar\omega n = \hbar\omega < n|a^\dagger a|n >$$
Now from our results above, we know that acting on $|n>$ a must lead exclusively to $|n-1>$ so $|n-1>$ is the only intermediate state that can be reached when we sandwich the identity between $a^\dagger$ and a. Cancelling the factor of $\hbar\omega$, we have
$$n = < n|a^\dagger|n-1 >< n-1|a|n >= |<n-1|a|n>|^2,$$
which gives
$$n = |<n-1|a|n>|^2.$$
Choosing the possible phase factor to be 1, we have
$$<n-1|a|n>= \sqrt{n} =< n|a^\dagger|n-1 >$$
Rewriting these, we have
$$a|n >= \sqrt{n}|n-1>, \quad a^\dagger|n >= \sqrt{(n+1)}|n+1>.$$
which gives
$$a^\dagger a|n >= n|n >$$
so $a^\dagger a$ is a number (of excitations) operator.

Let us show how to construct the states $|n>$ using $a^\dagger$. Writing out the first few states, we have
$$\begin{array}{rcl} a^\dagger|0 > & = & |1 > \\ a^\dagger|1 > & = & \sqrt{2}|2 > \\ a^\dagger|2 > & = & \sqrt{3}|3 > \\ \vdots & & \vdots \\ a^\dagger|n > & = & \sqrt{n+1}|n+1 > \end{array}$$
Using these equations to express everything in terms of $|0 >$, we have
$$|n >= \frac{1}{\sqrt{n!}}(a^\dagger)^n|0 >$$
We can use this last equation to find the wave functions for all the states. Using
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(X - i\frac{P}{m\omega}),$$
we have
$$< x|n >= \frac{1}{\sqrt{n!}}(\frac{m\omega}{2\hbar})^{n/2} < x|(X - i\frac{P}{m\omega})|0 >,$$
or
$$< x|n >= \frac{1}{\sqrt{n!}}(\frac{m\omega}{2\hbar})^{n/2}(x - \frac{\hbar}{m\omega}\partial_x) < x|0 > .$$
This formula will generate the explicit form for $< x|n >$ for any n.

Coherent States   Coherent states are an important class of states that can be realized by any system which can be represented in terms of a harmonic oscillator, or sums of harmonic oscillators. They are the answer to the question, what is the state of a quantum oscillator when it is behaving as classically as possible? As a practical example, the state of photons in a laser is quantum mechanical, but also behaves classically in many ways. Coherent states are relevant to this situation.

The definition of a coherent state $|\alpha>$ is that it is an eigenstate of a, the destruction operator, $a|\alpha> = \alpha|\alpha>$, with $<\alpha|\alpha> = 1$. From its definition, a is certainly not a self-adjoint operator, so it if has eigenstates at all, the eigenvalues need not be real, and in fact the value of α is any complex number.

Minimum Uncertainty   It is easy to show that coherent states exist, but let us first note one of their most characteristic properties, namely they have minimum Heisenberg uncertainty. Let us first express X and P in terms of a and $a^\dagger$. We have

$$
\begin{aligned}
X &= x_0(a + a^\dagger) \\
P &= -ip_0(a - a^\dagger)
\end{aligned}
$$

where

$$
\begin{aligned}
x_0 &= \sqrt{\frac{\hbar}{2m\omega}} \\
p_0 &= \sqrt{\frac{\hbar m\omega}{2}}
\end{aligned}
$$

Now consider various expected values, $<\alpha|X|\alpha>$, $<\alpha|X^2|\alpha>$, $<\alpha|P|\alpha>$, and $<\alpha|P^2|\alpha>$. First note that

$$
\begin{aligned}
<\alpha|a|\alpha> &= \alpha <\alpha|\alpha> \\
<\alpha|a^\dagger|\alpha> &= (<\alpha|a|\alpha>)^* = \alpha^* <\alpha|\alpha> .
\end{aligned}
$$

Finally, using $<\alpha|\alpha> = 1$, we have

$$
\begin{aligned}
<\alpha|X|\alpha> &= x_0(\alpha + \alpha^*) \\
<\alpha|P|\alpha> &= -ip_0(\alpha - \alpha^*)
\end{aligned}
$$

In calculating $<\alpha|X^2|\alpha>$ and $<\alpha|P^2|\alpha>$, the order of operators must treated with some care. We present the details for $<\alpha|X^2|\alpha>$. We have

$$
<\alpha|X^2|\alpha> = x_0^2 <\alpha|(a + a^\dagger)(a + a^\dagger)|\alpha> = x_0^2 <\alpha|(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger)|\alpha>
$$

Most of these evaluate directly. For example, $<\alpha|aa|\alpha> = \alpha^2 <\alpha|\alpha>$, $<\alpha|a^\dagger a|\alpha> = \alpha <\alpha|a^\dagger|\alpha> = \alpha\alpha^* <\alpha|\alpha>$, and $<\alpha|a^\dagger a^\dagger|\alpha> = (<\alpha|aa|\alpha>)^* = (\alpha^*)^2 <\alpha|\alpha>$. For

$< \alpha | a a^\dagger | \alpha >$, we use the commutator $[a, a^\dagger] = 1$ and rewrite it as $< \alpha | (a^\dagger a + 1) | \alpha >=$
$(\alpha \alpha^* + 1) < \alpha | \alpha >$. Finally, using $< \alpha | \alpha >= 1$, we have

$$< \alpha | X X | \alpha >= x_0^2 (\alpha \alpha + 2 \alpha \alpha^* + \alpha^* \alpha^* + 1) = x_0^2 \left((\alpha + \alpha^*)^2 + 1 \right)$$

From the results so far, we have

$$(\Delta X)^2 =< \alpha | X X | \alpha > - (< \alpha | X | \alpha >)^2 = x_0^2.$$

Entirely similar steps give

$$(\Delta P)^2 =< \alpha | P | \alpha > - (< \alpha | P | \alpha >)^2 = p_0^2.$$

Putting these together, we have

$$(\Delta X)(\Delta P) = x_0 p_0 = \frac{\hbar}{2}$$

which is the statement of minimum Heisenberg uncertainty. The steps just carried out have been done in the Schrödinger representation at $t = 0$. They go through equally well if we consider the uncertainties in X and P at an arbitrary time. This would be done by using Heisenberg opererators $X(t)$ and $P(t)$ in the above equations, which simply replace a by $a \exp(-i \omega t)$ and $a^\dagger$ by $a^\dagger \exp(i \omega t)$. The uncertainties in $X(t)$ and $P(t)$ remain x_0 and p_0, so minimum uncertainty continues to hold.

Existence   To demonstrate that coherent states exist, we first expand them in energy eigenstates of the harmonic oscillator,

$$|\alpha >= \sum_n |n >< n | \alpha > .$$

Using $a | \alpha >= \alpha | \alpha >$ and $a | n >= \sqrt{n} | n - 1 >$, we have

$$\sum_n \alpha | n >< n | \alpha >= \sum \sqrt{n} | n - 1 >< n | \alpha >= \sum_n \sqrt{n + 1} | n >< n + 1 | \alpha > .$$

Matching the coefficient of $|n >$ on both sides gives

$$\alpha < n | \alpha >= \sqrt{n + 1} < n + 1 | \alpha >$$

Applying this equation for $n = 0, 1, 2, \ldots$ we have

$$
\begin{array}{rclcl}
< 1 | \alpha > & = & \alpha < 0 | \alpha > & & \\
< 2 | \alpha > & = & \alpha < 1 | \alpha > / \sqrt{2} & = & \alpha^2 < 0 | \alpha > / \sqrt{2} \\
< 3 | \alpha > & = & \alpha < 2 | \alpha > / \sqrt{3} & = & \alpha^3 < 0 | \alpha > / \sqrt{3 \cdot 2} \\
\vdots & = & \vdots & = & \vdots
\end{array}
$$

The general result is then

$$< n|\alpha >= \frac{\alpha^n}{\sqrt{n!}} < 0|\alpha >$$

Demanding normalization, we have

$$1 =< \alpha|\alpha >= \sum_n | < n|\alpha > |^2 = \sum_n \frac{(|\alpha|^2)^n}{n!}| < 0|\alpha > |^2,$$

or

$$1 = \exp(|\alpha|^2)| < 0|\alpha > |^2.$$

Choosing a phase, we can set $< 0|\alpha >= \exp(-|\alpha|^2/2)$. We now have

$$|\alpha >= \exp(-\frac{|\alpha|^2}{2}) \sum_n \frac{\alpha^n}{\sqrt{n!}}|n >$$

This is a concrete formula for $|\alpha >$. Another, very convenient version is to make use of one of our previous formulas expressing $|n >$ in terms of repeated applications of $a^\dagger$. This gives

$$|\alpha >= \exp(-\frac{|\alpha|^2}{2}) \sum_n \frac{\alpha^n}{n!}(a^\dagger)^n|0 >= \exp(-\frac{|\alpha|^2}{2}) \exp(\alpha a^\dagger)|0 >$$

Level of Excitation Given a coherent state, we may ask how the level of excitation and average energy are related to α. Define the number operator $N = a^\dagger a$. When acting on an energy eigenstate, we have $N|n >= n|n >$. Taking the expected value of N in a coherent state, we obtain

$$< N >=< \alpha|N|\alpha >=< \alpha|a^\dagger a|\alpha >= \alpha < \alpha|a^\dagger|\alpha >= \alpha\alpha^*$$

This allows us to express α as

$$\alpha =< N > \exp(i\phi),$$

where ϕ is a phase angle. Now for the expected value of the Hamiltonian, we have

$$< H >=< \alpha|H|\alpha >= \hbar\omega(< N > +\frac{1}{2}) = \hbar\omega(\alpha\alpha^* + \frac{1}{2})$$

So given a certain average energy, the modulus of α is determined. To get insight into the meaning of the phase, consider the expected value of $X(t)$ in a coherent state. We have

$$< \alpha|X(t)|\alpha >= x_0(\alpha \exp(-i\omega t) + \alpha^* \exp(i\omega t) = x_0|\alpha|(\exp(-i(\omega t - \phi)) + \exp(i(\omega t - \phi)),$$

so $< X(t) >= x_0|\alpha|2\cos(\omega t - \phi)$. From this we can see that if $\phi = 0$, the oscillator expected value is that of a classical oscillator started at maximum displacement with

zero velocity, while if $\phi = \pi/2$, the expected value is that of a classical operator started at zero displacement with maximum velocity.

So far we see that in a coherent state, the value of $|\alpha|$ is in one-one correspondence with $< N >$ or $< H >$. However, a coherent state is certainly not anywhere near an energy eigenstate. There is considerable spread around the n-value determined by $< H >$. To quantify this consider P_n, the probability of a coherent state being found in the n-th eigenstate of the oscillator. We can write

$$P_n = | < n|\alpha > |^2 = \frac{(|\alpha|^2)^n}{n!} \exp(-|\alpha|^2)$$

Recalling that $< N >= |\alpha|^2$, we may write P_n as

$$P_n = \frac{< N >^n}{n!} \exp(- < N >),$$

which is a Poisson distribution. A well-known property of a Poisson distribution is that

$$(\Delta N)^2 =< N^2 > - < N >^2$$

is $< N >$. So there is considerable spread about $< N >$, but for large $< N >$, the average energy and level of excitation will appear to be well-defined. For example if $< N >= 100$, then $\Delta N = 10$. Thus an oscillator in a coherent state with a reasonably large $|\alpha|$, will appear to have a well-defined energy to a classical observer. Nevertheless many states of excitation near the one defined by $< N >$ are occupied.