1 Propagator and Path Integral

In general the quantity called the propagator is a matrix element of $U(t)$, and is often denoted by the letter K, so for a one particle system in one dimension, we would write

$$K(x,x',t) =< x|\exp(-\frac{i}{\hbar}Ht)|x' >$$

The first case to look at is a free particle,with Hamiltonian

$$H = \frac{P^2}{2m}$$

For this case we add a subscript f to denote that we are discussing a free particle, so here

$$K_f(x,x',t) = \int dp < x|\exp(-\frac{i}{\hbar}Ht)|p < p|x' >$$
$$= \int \frac{dp}{2\pi\hbar}\exp(\frac{i}{\hbar}p(x-x'))\exp(-\frac{i}{\hbar}\frac{p^2t}{2m}),$$

where we sandwiched the identity inside the matrix element. This is a Gaussian integral, which can be put in a standard form by re-arranging the quantity in the exponent. This move is called "completing the square." We have for the exponent,

$$-i[\frac{t}{2m\hbar}[(p-\frac{m(x-x')}{t})^2 - (\frac{m(x-x')}{t})^2]].$$

We define a new variable of integration

$$p' = p - m\frac{(x-x')}{t},$$

and integrate over p'. A standard Gaussian integral is

$$\int_{-\infty}^{+\infty}\exp(-i\alpha\frac{x^2}{2})dx = \sqrt{\frac{2\pi}{i\alpha}}.$$

Using this, we have an explicit formula for the propagator for a free particle,

$$K_f(x,x',t) = \sqrt{\frac{m}{2\pi i\hbar t}}\exp(\frac{i}{\hbar}\frac{m(x-x')^2}{2t})$$

The quantity in the exponent is a purely classical quantity once the factor of $i/\hbar$ is extracted. It has the same dimensions as $\hbar$, i.e. *action*, and in fact is the classical action for a free particle to go from x' to x in time t.

$$S_c(x,x',t) = \frac{m(x-x')^2}{2t}$$

Classically the particle simply travels in a straight line, starting from x' and ending at x.

$$x(t') = x' + vt'.$$

Putting $t' = t$, we have

$$x(t) = x = x' + vt.$$

The velocity is constant, and its magnitude is determined by the endpoint coordinates,

$$v = (\frac{x - x'}{t})$$

For a free particle, the Lagrangian is

$$L = \frac{m}{2}(\frac{dx}{dt})^2$$

The action is defined as the integral of the Lagrangian over time. The action can be computed for any "path" or function $x(t')$. The *classical action*, denoted as S_c is the action evalutated for the actual classical trajectory or path connecting x' and x. For our case, we have

$$S_c(x, x', t) = \int dt' L(x(t')) = \int dt' \frac{m}{2}(\frac{dx}{dt'})^2 = \frac{m(x - x')^2}{2t}$$

The propagator itself has no physical state in either its bra or ket slot. When folded into a physical state, the propagator does what its name implies, namely moves a state in time. For some initial state $|\Psi>$, the propagator will give the state at time t;

$$< x|\Psi(t) >= \int dx' < x| \exp(-\frac{i}{\hbar}Ht)|x' >< x'|\Psi >= \int dx' K(x, x', t) < x'|\Psi >$$

Path Integral Let us now turn to a case of a particle interacting with a potential. We remain in one dimension, since all the ideas to be presented have an easy generalization to higher dimensions. Suppose the Hamiltonian of our system is

$$H = \frac{P^2}{2m} + V(X)$$

We are interested in computing

$$K(x, x', t) =< x| \exp(-\frac{i}{\hbar}Ht)|x' >$$

just as before. This is not directly computable for a general potential, so we start splitting the time interval up. So for one split, we have

$$< x| \exp(-\frac{i}{\hbar}Ht)|x' >=< x| \exp(-\frac{i}{\hbar}Ht_1) \exp(-\frac{i}{\hbar}Ht_2)|x' >$$

where $t = t_1 + t_2$. This is not obviously helpful. However, we go on and make more splits. The physical idea of path integral is that as we split up the time into smaller and smaller slices, the matrix elements become very simple. So let us insert a whole sequence of times, equally spaced by a small amount ϵ, where the number of intervals is given by $t = N\epsilon$. Writing K for this case, we have

$$K(x, x', t) =< x|\underbrace{e^{-\frac{i\epsilon H}{\hbar}} e^{-\frac{i\epsilon H}{\hbar}} \dots e^{-\frac{i\epsilon H}{\hbar}}}_{\text{N factors}}|x' >$$

It is still not obvious why this is useful. But let us continue. Since we are in coordinate space, it makes sense to insert the identity in terms of coordinates between each of the factors. This leads us to

$$K(x, x', t) = \int dx_1 \dots dx_{N-1} < x|e^{-\frac{i\epsilon H}{\hbar}}|x_{N-1} >< x_{N-1}|e^{-\frac{i\epsilon H}{\hbar}}|x_{N-2} > \dots < x_1|e^{-\frac{i\epsilon H}{\hbar}}|x' >$$

The point of the path integral approach is that as $\epsilon \to 0$, the matrix elements become very simple. While in general it is wrong to split operators apart in the exponent, since both the kinetic energy and potential are multiplied by the small parameter ϵ, it is valid here to $O(\epsilon)^2$. In general with operators we can write

$$\exp(A)\exp(B) = \exp(C),$$

but the formula for C is not just $A + B$. The fact that C exists is called the Baker-Haussdorf theorem. The only part of it that we need right now is the fact that

$$C = A + B + \frac{1}{2}[A, B] + \dots,$$

where the additional terms involve multiple commutators, such as $[A, [A, B]]$, etc. If both operators have a factor of ϵ, the terms involving commutators are $O(\epsilon)^2$ and can be dropped. We then have

$$< x_k|e^{-\frac{i\epsilon H}{\hbar}}|x_{k-1} >\approx< x_k|e^{-\frac{i\epsilon P^2}{\hbar 2m}}|x_{k-1} >< x_k|e^{-\frac{i\epsilon V(X)}{\hbar}}|x_{k-1} >$$

The first factor is a free propagator, the the second factor can be evaluated by replacing the operator X, but either x_{k-1} or x_k. For small ϵ the difference between these two choices is again of $O(\epsilon)^2$. We will write our formulas for X replaced by x_{k-1}. We finally have then

$$< x_k|e^{-\frac{i\epsilon H}{\hbar}}|x_{k-1} >\to \sqrt{\frac{m}{2\pi i\hbar\epsilon}} \exp\left(\frac{i}{\hbar}(\frac{m(x_k - x_{k-1})^2}{2\epsilon} - \epsilon V(x_k))\right)$$

Adding up the exponents for each interval, the net exponenent is

$$\sum_{k=1}^{N}\left(\frac{m(x_k - x_{k-1})^2}{2\epsilon} - \epsilon V(x_k)\right) \approx \int_0^t L(x(t'))dt',$$

where on the right side of the equation, we identify the total exponent as the time integral of the Lagrangian. This integral is the *action*, so in the path integral approach, every path between the endpoints is considered, and each path is weighted by its action. Symbolically, the path integral for the propagator is usually written as

$$K(x,x',t) =< x|\exp(-\frac{i}{\hbar}Ht)|x' >= \int \exp(\frac{i}{\hbar}S_c(x(t))\mathcal{D}(x(t))$$

where the measure of integration is well defined once the time interval is broken into small pieces. In that case we have

$$\mathcal{D}(x(t')) = (\sqrt{\frac{m}{2\pi i\hbar\epsilon}})^N dx_1 \dots dx_{N-1}$$