## Charge in Magnetic Field

Consider the motion of a charge in combined electric and magnetic fields. The equation of motion is

$$
\begin{equation*}
m \frac{d \vec{v}}{d t}=q \vec{E}+\frac{q}{c} \vec{v} \times \vec{B}, \tag{1}
\end{equation*}
$$

where in Gaussian units, we have

$$
\vec{E}=-\frac{1}{c} \frac{\partial \vec{A}}{\partial t}-\vec{\nabla} \phi, \quad \vec{B}=\vec{\nabla} \times \vec{A},
$$

where $\phi, \vec{A}$, are scalar and vector potentials, respectively. In order to do quantum mechanics, a classical Lagrangian and Hamiltonian are needed. The classical Lagrangian is

$$
L=\frac{m}{2} \vec{v} \cdot \vec{v}+\frac{q}{c} \vec{v} \cdot \vec{A}-q \phi .
$$

It is easily checked that the Lagrange equations from $L$ produce Eq.(1). The canonical momentum from the Lagrangian is

$$
\vec{p}=m \vec{v}+\frac{q}{c} \vec{A},
$$

and the classical Hamiltonian is constructed in the usual way;

$$
H=\vec{p} \cdot \vec{v}-L
$$

Using

$$
\vec{v}=\frac{1}{m}\left(\vec{p}-\frac{q}{c} \vec{A}\right),
$$

we have

$$
H=\frac{1}{2 m}\left(\vec{p}-\frac{q}{c} \vec{A}\right)^{2}+q \phi .
$$

To proceed to quantum mechanics, we replace $\vec{p}$ by $\hbar \vec{\nabla} / i$ as usual, and obtain

$$
H=\frac{1}{2 m}\left(\frac{\hbar}{i} \vec{\nabla}-\frac{q}{c} \vec{A}\right)^{2}+q \phi .
$$

The description of a given magnetic field by a vector potential is not unique. We will use a so-called "symmetric" gauge, where in terms of an applied constant magnetic field, $\vec{A}$ is

$$
\vec{A}=\frac{1}{2} \vec{B} \times \vec{r}
$$

Using this vector potential and writing the Hamiltonian for a hydrogen atom in an external magnetic field, we have

$$
H=-\frac{\hbar^{2} \nabla^{2}}{2 m}+\frac{e \hbar}{2 m c} \vec{B} \cdot(\vec{L}+2 \vec{S})+\frac{e^{2}}{8 m c^{2}}(\vec{B} \times \vec{r})^{2}-\frac{e^{2}}{r},
$$

where the charge of the electron is $-|e|$, and the contribution of the electron magnetic moment has been included. (Angular momenta are measured in units of $\hbar$ in this formula.) The term quadratic in the magnetic field is a new term, not seen in our previous treatment, where we used simple classical arguments to get the magnetic moment of the circulating charge in an atom. In fact, the quadratic term in $B$ is very small for an atom. The reason is that the Coulomb potential keeps the electron at distances of order $a_{0}$ from the nucleus, and for those distances, it is easily seen that the quadratic term is very small compared to the linear term. To show this, the following numerical formula is useful:

$$
e(\text { gauss }) a_{0}=1.6 \times 10^{-6} \mathrm{eV}
$$

Free Charge in Magnetic Field We now turn away from electrons in atoms, to the quantum mechanics of a free charge in an external magnetic field. Although such a charge is typically an electron, to avoid excessive minus signs, we take the charge to be positive, $q=+|e|$. For this case the Hamiltonian is

$$
q=|e|, \quad H=\frac{1}{2 m}\left(\frac{\hbar}{i} \vec{\nabla}-\frac{e}{c} \vec{A}\right)^{2}
$$

For an external magnetic field along $\hat{z}$, we have

$$
\vec{B}=B \hat{z}, \quad A_{y}=\frac{1}{2} B x, \quad A_{x}=-\frac{1}{2} B y,
$$

so the Hamiltonian is

$$
H=\frac{1}{2 m}\left(-\hbar^{2} \nabla^{2}-\frac{e B}{c} L_{z}+\frac{e^{2} B^{2}}{4 e^{2}}\left(x^{2}+y^{2}\right)\right) .
$$

To avoid repeated writing of physical constants, we rewrite $H$ in "magnetic units." There is a natural unit of length, known as the magnetic length, as well as a natural frequency, the cyclotron frequency. The definitions are

$$
l_{B}=\sqrt{\frac{\hbar}{m \omega_{c}}}, \quad \omega_{c}=\frac{e B}{m c}, \quad l_{B}=a_{0} \frac{4.83 \times 10^{4}}{\sqrt{B(\text { gauss })}}
$$

The last formula shows that for any magnetic field $\leq 1 T$, the magnetic length is many Bohr radii. This is ultimately the reason the quadratic term in $B$ must be kept in this case, whereas it was small for an electron in an atom. Using $l_{B}$ as the unit of length, $\hbar \omega_{c}$ as the unit of energy, and measuring $L_{z}$ in units of $\hbar$, the Hamiltonian now reads

$$
H=-\frac{1}{2}\left(\nabla^{2}+L_{z}\right)+\frac{\rho^{2}}{8}
$$

where $\rho^{2}+x^{2}+y^{2}$. This Hamiltonian looks rather simple. It becomes even simpler if we study the special case,

$$
L_{z}=0 .
$$

Then we have

$$
H=-\frac{\partial_{x}^{2}+\partial_{y}^{2}}{2}+\frac{x^{2}+y^{2}}{8},
$$

which clearly is just two harmonic oscillators, one in $x$, the other in $y$. The ground state wave function and eigenvalue are easily found;

$$
\Psi_{0,0}=\frac{1}{\sqrt{2 \pi}} \exp \left(-\frac{\rho^{2}}{4}\right), \quad E=\frac{1}{2}
$$

Now let us relax the condition $L_{z}=0$ and search for some more wave functions and eigenvalues. As in all oscillator problems we expect the exponential of the ground state to appear in excited state wave functions. A simple trial wave function with $L_{z}=m$ is then

$$
\Psi=\rho^{|m|} \exp (i m \phi) \exp \left(-\frac{\rho^{2}}{4}\right)
$$

(NOTE: The variable $\phi$ is the angle in plane polar coordinates here.) It is best now to use plane polar coordinates in $H$, and we have

$$
H=-\frac{1}{2}\left(\frac{1}{\rho} \partial_{\rho} \rho \partial_{\rho}+\frac{1}{\rho^{2}} \partial_{\phi}^{2}+\frac{1}{i} \partial_{\phi}\right)+\frac{\rho^{2}}{8}
$$

Applying $H$ in this form to our trial wave function, we find that it is indeed an eigenfunction

$$
H \Psi_{m}=\left(\frac{1+|m|-m}{2}\right) \Psi_{m},
$$

So the eigenvalue depends on the sign of $m$ as follows: for

$$
m>0, \quad E=\frac{1}{2}
$$

while for

$$
m<0, \quad E=|m|+\frac{1}{2}
$$

Let us relate these eigenvalues to the classical motion of this charge. The classical motion is a circle, and the positive charge moves clockwise around the $B$ field, or in other words the classical motion corresponds to $m<0$. The states with $m>0$ or the "wrong" value of angular momentum, we see have the minimum possible energy, $E=1 / 2$, the same as $\Psi_{0,0}$. So the lowest energy state is tremendously degenerate; any value of $m \geq 0$ has $E=1 / 2$. It turns out that the excited states are also tremendously degenerate as well. This will become clear later.

For the case $m<0$, the wave function has the properties appropriate to clockwise circular motion, centered on the origin. What we have found is only a small subset of the excited states. In particular, we at present cannot see how to represent circular motion which is centered at some place other than the origin. For that we turn to creation and destruction operator methods.

Creation and Destruction Operators We will temporarily introduce a frequency, so as to make the formulas more familiar. For us $\omega=1 / 2$, where the unit of frequency is the cyclotron frequency, $\omega_{c}=e B / m c$. We retain the $\hbar=1$ feature of magnetic units. We write

$$
H=H_{0}-\omega L_{z},
$$

where

$$
H_{0}=\frac{1}{2}\left(p_{x}^{2}+\omega^{2} x^{2}+p_{y}^{2}+\omega^{2} y^{2}\right)
$$

Clearly $H_{0}$ is just two independent harmonic oscillators. Introducing creation and destruction operators for $x$ and $y$, we have

$$
x=\frac{a_{x}+a_{x}^{\dagger}}{\sqrt{2 \omega}}, \quad y=\frac{a_{y}+a_{y}^{\dagger}}{\sqrt{2 \omega}}
$$

and

$$
p_{x}=-i \sqrt{\frac{\omega}{2}}\left(a_{x}-a_{x}^{\dagger}\right), \quad p_{y}=-i \sqrt{\frac{\omega}{2}}\left(a_{y}-a_{y}^{\dagger}\right) .
$$

Inverting these equations, we obtain

$$
\begin{equation*}
a_{x}=\frac{1}{\sqrt{2 \omega}}\left(\omega x+i p_{x}\right), \quad a_{y}=\frac{1}{\sqrt{2 \omega}}\left(\omega y+i p_{y}\right) . \tag{2}
\end{equation*}
$$

Using these results, we can express $H_{0}$ and $L_{z}$ in terms of creation and destruction operators,

$$
H_{0}=\frac{\omega}{2}\left(a_{x}^{\dagger} a_{x}+\frac{1}{2}+a_{y}^{\dagger} a_{y}+\frac{1}{2}\right),
$$

and

$$
L_{z}=-i\left(a_{x}^{\dagger} a_{y}-a_{y}^{\dagger} a_{x}\right)
$$

Charged Oscillators As always in planar problems, it is better to use complex combinations rather than Cartesian components. We define

$$
\begin{equation*}
\Phi=x+i y, \quad \Phi^{\dagger}=x-i y . \tag{3}
\end{equation*}
$$

Using our results above, we can write $\Phi$ as

$$
\Phi=\frac{1}{\sqrt{2 \omega}}\left(\left(a_{x}+i a_{y}\right)+\left(a_{x}^{\dagger}+i a_{y}^{\dagger}\right)\right) .
$$

This is still using Cartesian operators. The so-called "charged" operators are defined by setting

$$
\begin{equation*}
\Phi=\frac{a+b^{\dagger}}{\sqrt{\omega}} . \tag{4}
\end{equation*}
$$

(NB The term "charged" is slightly out of place here. It is not the charge of the particle, rather in our present problem it is $-L_{z}$. The terminology has a field theory origin.) Comparing the two forms for $\Phi$, we have

$$
\begin{equation*}
a=\frac{a_{x}+i a_{y}}{\sqrt{2}}, \quad b=\frac{a_{x}-i a_{y}}{\sqrt{2}} . \tag{5}
\end{equation*}
$$

The $a$ 's and $b$ 's form a set of independent creation and destruction operators.

$$
\left[a, a^{\dagger}\right]=\left[b, b^{\dagger}\right]=1, \quad[a, b]=\left[a, b^{\dagger}\right]=0
$$

The goal now is to express our Hamiltonian in the $a, b$ language. For this we invert the definitions of $a$ and $b$ to find

$$
a_{x}=\frac{1}{\sqrt{2}}(a+b), \quad a_{y}=-i \frac{1}{\sqrt{2}}(a-b)
$$

Substituting for these in $H_{0}$, we have

$$
H_{0}=\omega\left(a^{\dagger} a+\frac{1}{2}+b^{\dagger} b+\frac{1}{2}\right) .
$$

Doing the same for $L_{z}$, gives

$$
L_{z}=\left(b^{\dagger} b-a^{\dagger} a\right)
$$

Restoring

$$
\omega=\frac{1}{2}
$$

we finally have

$$
H=H_{0}-\omega L_{z}=a^{\dagger} a+\frac{1}{2}
$$

This is an interesting formula. We see that the energy is independent of the $b$ oscillators and only depends on the $a$ oscillators. So a state like

$$
\left(a^{\dagger}\right)^{n_{2}}\left(b^{\dagger}\right)^{n_{1}} \mid 0,0>
$$

has energy $E=n_{2}+1 / 2$, but angular momentum $L_{z}=n_{1}-n_{2}$. This allows us to see that excited states are also highly degenerate. The value of $n_{2}$ determines the energy, while $n_{1}$ can have any positive value whatsoever.

For finding explicit forms of wave functions, it is useful to have the $a$ 's and $b$ 's expressed as differential operators. Using Eqs.(2) and (5), we can do this. The results are

$$
\begin{array}{ll}
a=\frac{1}{\sqrt{2}}\left(\frac{x+i y}{2}+\left(\partial_{x}+i \partial_{y}\right)\right), \quad a^{\dagger}=\frac{1}{\sqrt{2}}\left(\frac{x-i y}{2}-\left(\partial_{x}-i \partial_{y}\right)\right) \\
b=\frac{1}{\sqrt{2}}\left(\frac{x-i y}{2}+\left(\partial_{x}-i \partial_{y}\right)\right), \quad b^{\dagger}=\frac{1}{\sqrt{2}}\left(\frac{x+i y}{2}-\left(\partial_{x}+i \partial_{y}\right)\right)
\end{array}
$$

The best way to use these operators is to express them in terms of $x_{+}=x+i y, x_{-}=x-i y$, and $\partial_{+}=\partial_{x}+i \partial_{y}$, and $\partial_{-}=\partial_{x}-i \partial_{y}$.

Coherent States and Classical Motion The best way to see the classical motion, including the location of the center of the circle is to use coherent states. Our problem involves two sorts of creation and destruction operators, $a, a^{\dagger}$, and $b, b^{\dagger}$. It will turn out that the $a$ operators control the circular motion, and the $b$ operators control the location of the center of the circle. We will use states which are coherent in both $a$ and $b$. That is we have states $\mid \alpha, \beta>$ with the properies

$$
a|\alpha, \beta>=\alpha| \alpha, \beta>, \quad b|\alpha, \beta>=\beta| \alpha, \beta>.
$$

These have the important properties

$$
\begin{array}{ll}
<\alpha, \beta|\alpha| \alpha, \beta>=\alpha, & <\alpha, \beta\left|\alpha^{\dagger}\right| \alpha, \beta>=\alpha^{*} \\
<\alpha, \beta|\beta| \alpha, \beta>=\beta, & <\alpha, \beta\left|\beta^{\dagger}\right| \alpha, \beta>=\beta^{*}
\end{array}
$$

All of our coherent states are built by using $\mid 0,0>$ which has wave function

$$
<x, y \mid 0,0>=\frac{1}{\sqrt{2 \pi}} \exp \left(-\frac{x^{2}+y^{2}}{4}\right)
$$

The coherent state $\mid \alpha, \beta>$ is built by applying exponentials of $a^{\dagger}$ and $b^{\dagger}$. We have

$$
\left|\alpha, \beta>=\exp \left(\alpha a^{\dagger}+\beta b^{\dagger}\right)\right| 0,0>\exp \left(-\frac{1}{2}\left(|\alpha|^{2}+|\beta|^{2}\right)\right.
$$

To relate these to expected values of $x$ and $y$, we return to Eqs.(3)and (4) and solve for $x$ and $y$. This gives

$$
x=\frac{1}{2 \sqrt{\omega}}\left(a+a^{\dagger}+b+b^{\dagger}\right), \quad y=\frac{1}{2 i \sqrt{\omega}}\left(a-a^{\dagger}-b+b^{\dagger}\right) .
$$

The classical motion will be the result of calculating

$$
<\alpha, \beta|x(t)| \alpha, \beta>, \quad<\alpha, \beta|y(t)| \alpha, \beta>
$$

where

$$
x(t)=\exp (i H t) x \exp (-i H t), \quad y(t)=\exp (i H t) y \exp (-i H t)
$$

From our previous results, we have

$$
H=2 \omega\left(a^{\dagger} a+1\right)
$$

From this formula it follows that the operators $b$ and $b^{\dagger}$ are independent of time, while

$$
a(t)=\exp (-2 i \omega t) a, a^{\dagger}(t)=\exp (2 i \omega t) a^{\dagger}
$$

Since $2 \omega$ is just the cyclotron frequency, we see that $a(t)$ and $a^{\dagger}(t)$ move with the cyclotron frequency. These operators describe the circular motion. Meanwhile the center of the circle, which is constant in time is determined by the $b$ and $b^{\dagger}$ operators. Denoting the center of the circle by $x_{0}, y_{0}$, we have

$$
x_{0}=\frac{1}{2 \sqrt{\omega}}\left(\beta+\beta^{*}\right), \quad y_{0}=-\frac{1}{2 i \sqrt{\omega}}\left(\beta-\beta^{*}\right),
$$

where as before $\omega=1 / 2$, in units of $\omega_{c}$.

Changing the Gauge We have so far done all our calculations in the so-called "symmetric" gauge, which has

$$
\vec{A}=\frac{1}{2} \vec{B} \times \vec{r} .
$$

All physical quantities can be calculated in this gauge. A gauge must be chosen, but once a gauge is chosen, every physical quantity can be determined. From this viewpoint, there is no need to discuss other gauges. Nevertheless, certain quantities are easier to obtain in different gauges, and it is imporant to know how to change the gauge, which is nothing other than changing $\vec{A}$ in such a way that the electromagnetic fields $\vec{E}$ and $\vec{B}$ remain the same. Suppose then that we want to go from $\vec{A}_{1}$ to $\vec{A}_{2}$. To maintain the same fields, the two must differ by at most a gradient,

$$
\vec{A}_{2}(\vec{x})=\vec{A}_{1}(\vec{x})+\vec{\nabla} \chi(\vec{x})
$$

Wave functions transform when a gauge transformation is made:

$$
\Psi_{2}(\vec{x})=\exp \left(\frac{i q \chi}{\hbar c}\right) \Psi_{1}(\vec{x}) .
$$

Gauge transforming both the vector potential and the wave function, the Schrödinger operator

$$
\frac{1}{2 m}\left(\frac{\hbar \vec{\nabla}}{i}-\frac{q \vec{A}_{2}}{c}\right)^{2} \Psi_{2}
$$

goes over to

$$
\begin{aligned}
& \frac{1}{2 m}\left(\frac{\hbar \vec{\nabla}}{i}-\frac{q \vec{A}_{1}+\vec{\nabla} \chi}{c}\right)^{2} \exp \left(\frac{i q \chi}{\hbar c}\right) \Psi_{1} \\
& \quad=\exp \left(\frac{i q \chi}{\hbar c}\right) \frac{1}{2 m}\left(\frac{\hbar \vec{\nabla}}{i}-\frac{q \vec{A}_{1}}{c}\right)^{2} \Psi_{1}
\end{aligned}
$$

As a consequence, if we start with a $\Psi_{2}$ eigenstate and make a gauge transformation, The Schrödinger equation

$$
\frac{1}{2 m}\left(\frac{\hbar \vec{\nabla}}{i}-\frac{q \vec{A}_{2}}{c}\right)^{2} \Psi_{2}=E \Psi_{2}
$$

becomes

$$
\exp \left(\frac{i q \chi}{\hbar c}\right)\left(\frac{1}{2 m}\left(\frac{\hbar \vec{\nabla}}{i}-\frac{q \vec{A}_{1}}{c}\right)^{2} \Psi_{1}\right)=\exp \left(\frac{i q \chi}{\hbar c}\right) E \Psi_{1}
$$

So the wave functions transform by a space-dependent phase factor, while the energy is gauge invariant.

Landau Gauge Many papers written on the quantum mechanics of free charges in magnetic fields use a gauge introduced by Landau, where

$$
A_{x}=-B y, A_{y}=0, A_{z}=0
$$

Regarding the symmetric gauge as $\overrightarrow{A_{1}}$ and Landau gauge as $\overrightarrow{A_{2}}$, we can determine the gauge function $\chi$ by writing

$$
(-B y, 0,0)=\left(-\frac{B y}{2}, \frac{B x}{2}, 0\right)+\vec{\nabla} \chi
$$

which gives

$$
\vec{\nabla} \chi=\left(-\frac{B y}{2},-\frac{B x}{2}, 0\right)
$$

or discarding a constant,

$$
\chi=-\frac{B x y}{2} .
$$

Thus for a charge $|e|$, we would transform wave functions as follows:

$$
\Psi_{L}=\exp \left(-i \frac{e}{2 c} B x y\right) \Psi_{1}
$$

The Schrödinger equation in Landau gauge is

$$
\begin{equation*}
\frac{1}{2 m}\left(\left(\frac{\hbar}{i} \partial_{x}+\frac{e B y}{c}\right)^{2}+\left(\frac{\hbar}{i} \partial_{y}\right)^{2}\right) \Psi=E \Psi \tag{6}
\end{equation*}
$$

where as in the symmetric gauge, we have restricted the motion to the $x-y$ plane. Going over to magnetic units as we did in the symmetric gauge, the Schrödinger equation in Landau gauge reads

$$
\frac{1}{2}\left(\left(\frac{1}{i} \partial_{x}+y\right)^{2}+\left(\frac{1}{i} \partial_{y}\right)^{2}\right) \Psi=E \Psi
$$

where as usual distances are measured in units of $l_{B}=\sqrt{\hbar / m \omega_{c}}$ and energy is measured in units of $\hbar \omega_{c}$. The Schrödinger equation in Landau gauge, Eq.(6) does not involve an $x$-dependent potential, so plane wave dependence on $x$ can be assumed. We set

$$
\Psi=\frac{1}{\sqrt{2 \pi}} e^{i k x} \tilde{\Psi}(y)
$$

The equation for $\tilde{\Psi}$ is

$$
\left(-\frac{\partial_{y}^{2}}{2}+\frac{(y+k)^{2}}{2}\right) \tilde{\Psi}=E \tilde{\Psi}
$$

which is just a displaced harmonic oscillator. If we put the oscillator in its ground state, we have

$$
\Psi(x, y)=\frac{1}{2 \pi} e^{i k x} e^{-\frac{1}{2} y^{2}}
$$

which has $E=1 / 2$. Again we see the huge degeneracy of a given energy eigenvalue. Any value of $k$ gives the same energy.

Degeneracy of Landau levels An important feature of the levels of a charge in a magnetic field is the number of particles which can be put into a given level, where of course we are referring to fermions, electrons in almost all cases. The Pauli principle allows only one particle per state. We have seen that the levels are quite degenerate. It is fair to ask, if there is a rectangular sample of dimensions $L_{x} \times L_{y}$, how many electrons can be put into a given level with energy

$$
E=\hbar \omega_{c}\left(n+\frac{1}{2}\right) ?
$$

The answer is independent of gauge, but is easy to find in Landau gauge. The restriction to

$$
0 \leq x \leq L_{x}
$$

makes the allowed values of $k$ discrete, given by

$$
k=\frac{2 n \pi}{L_{x}}
$$

So we can start off counting states by forming the sum

$$
\sum_{k}
$$

which counts each state once. This has no apparent restriction on $k$ until we note that the plane wave states in $x$ are accompanied by the harmonic oscillator wave function of $y+k$. For any state, not just the ground state, this will have the gaussian factor

$$
\exp \left(-\frac{1}{2}(y+k)^{2}\right)
$$

Keeping the center of this gaussian inside the range $0 \leq y \leq L_{y}$, will restrict the range of $k$. For $y=0$, the center of the gaussian is at $k=0$, while for $y=L_{y}$, the center of the gaussian is at $k=-L_{y}$. The range of $k$ values must then lie in the range $-L_{y} \leq k \leq 0$. Returning to the counting of states, we have

$$
\sum_{k} \rightarrow \frac{L_{x}}{2 \pi} \int_{-L_{y}}^{0} d k=\frac{L_{x} L_{y}}{2 \pi}
$$

Returning to ordinary units, the degeneracy is

$$
\frac{L_{x} L_{y}}{2 \pi l_{B}^{2}}=\left(L_{x} L_{y} B\right) \frac{e}{2 \pi \hbar c}
$$

We note that $L_{x} L_{y} B$ is just the magnetic flux through the area $L_{x} L_{y}$. The quantity

$$
\Phi_{0}=\frac{2 \pi \hbar c}{2 e}=2.0678 \times 10^{-7} \text { Gauss } \cdot \mathrm{cm}^{2}
$$

also has dimensions of flux and is known as the "quantum fluxoid." Note the factor of $2 e$ in the definition. Counting both components of spin, a given Landau level can accommodate

$$
\frac{L_{x} L_{y} B}{\Phi_{0}}
$$

electrons. In an experiment, the dimensions $L_{x}, L_{y}$ are known only with limited precision. Further, the degeneracy as calculated above is not a precise number. The formula gives the term in the degeneracy which is proportional to the area of the sample. There would of course be correction terms which grow more slowly with the dimensions of the sample. Nevertheless, the filling of Landau levels can be seen experimentally by examining physical quantities such as the magnetization as the magnetic field is varied. Certain quantities vary periodically in the magnetic field and this can be used as an imporant diagnostic of the system being studied.

