Green Functions in Many Body Quantum Mechanics

NOTE This section contains some advanced material, intended to give a brief introduction to methods used in many body quantum mechanics. The material at the end of this section (beyond $\tilde{\Sigma}^{(1)}$) will not be covered on future homework or the final exam.

Many Fermion Systems The formalism in this section is designed to handle uniform systems with many fermions. As in any quantum system, the first object of interest is the properties of the ground state. However, when the system contains a macroscopic number of particles, the wave function of the ground state contains too much information to be useful. Instead what is studied are matrix elements of simple operators in the ground state. Our system is N fermions in a volume V, with N/V finite. The fermions could be nucleons, He^3 atoms, or electrons. We introduce two notations:

$$\text{true ground state}: \quad |\Psi_N^{(0)}>$$

$$\text{free ground state}: \quad |\Phi_N^{(0)}>$$

The free ground state is rather simple. It consists of a filled "Fermi sea" of spin 1/2 fermions where the Fermi wave vector k_F is determined by the density of the system,

$$\frac{N}{V} = \frac{2}{V}\sum_{\vec{k}}(1) = \frac{2}{(2\pi)^3}\int_0^{k_F} d\vec{k} = \frac{k_F^3}{3\pi^2}$$

In a so-called "normal" Fermi system, a good understanding of the true ground state $|\Psi_N^{(0)}>$ can be obtained by doing perturbation theory (perhaps to all orders) starting from the free ground state, $|\Phi_N^{(0)}>$. This section contains an introduction to the use of perturbation theory methods for many fermion systems.

The Propagator As mentioned above, the actual wave function of the ground state contains too much information, and the objects of interest instead are correlation functions of fields (or products of fields) at different space-time points. The simplest such object is called the "propagator" or the "single particle Green function" and is defined as follows:

$$G_{s,s'}(\vec{x},t;\vec{x'},t') = -\frac{i}{\hbar} < \Psi_N^{(0)}|T[\hat{\Psi}_s(\vec{x},t)\hat{\Psi}^\dagger_{s'}(\vec{x'},t')]|\Psi_N^{(0)}>$$

The fields in $G_{s,s'}(\vec{x},t;\vec{x'},t')$ are Heisenberg operators,

$$\hat{\Psi}_s(\vec{x},t) = \exp(\frac{i\hat{H}t}{\hbar})\hat{\Psi}_s(\vec{x},0)\exp(\frac{-i\hat{H}t}{\hbar}),\ \hat{\Psi}^\dagger_s(\vec{x},t) = \exp(\frac{i\hat{H}t}{\hbar})\hat{\Psi}^\dagger_s(\vec{x},0)\exp(\frac{-i\hat{H}t}{\hbar}),$$

where $\hat{H}$ is the full Hamiltonian of the system. The T or time ordering symbol is defined as follows. For any two operators $\hat{A}(t)$ and $\hat{B}(t')$, we define

$$T(\hat{A}(t)\hat{B}(t')) \equiv \left\{ \begin{array}{cc} \hat{A}(t)\hat{B}(t') & t > t' \\ (-1)^F\hat{B}(t')\hat{A}(t) & t' > t \end{array} \right. ,$$

where in $(-1)^F$, F is the number of Fermi operator interchanges. For our case where both A and B are Fermi operators, $F = 1$. The introduction of this phase is needed because certain operators (Fermi operators) obey anti-commutation relations, whereas others (Bose operators) obey commutation relations. The vector potential studied earlier is a Bose operator. It should be noted that the T symbol is a *definition.* It is not in any way ignoring commutators or anti-commutators. The importance of using the T or time ordering operation was emphasized by F.J. Dyson, after Feynman emphasized the importance of propagators and introduced his famous diagrams.

The propagator of a Fermi system contains information about the "single particle spectrum" sometimes called the "quasi-particle spectrum." This is a particularly useful idea for excitations with wave numbers near k_F. For a normal Fermi system, the spectrum of such quasiparticles dominates many of the properties of the system. As will be seen below, the quasiparticle spectrum is determined by the poles in the frequency plane of the Fourier transform of the propagator. From a space-time viewpoint, $G_{s,s'}(\vec{x}, t; \vec{x'}, t')$ can be visualized as follows: a particle is added to the system at $\vec{x'}, t'$. Then $G_{s,s'}(\vec{x}, t; \vec{x'}, t')$ measures the amplitude for that particle to arrive at $\vec{x}, t$ where it is removed from the system.

Free Propagator For a normal Fermi system, a perturbative treatment begins with the free propagator, defined by

$$G^{(0)}_{s,s'}(\vec{x}, t; \vec{x'}, t') = -\frac{i}{\hbar} < \Phi^{(0)}_N | T[\hat{\Psi}_s(\vec{x}, t)\hat{\Psi}^\dagger_{s'}(\vec{x'}, t')] | \Phi^{(0)}_N >, \tag{1}$$

where the field operators now move in time with the free Hamiltonian,

$$\hat{\Psi}_s(\vec{x}, t) = \exp(\frac{i\hat{H}_0 t}{\hbar})\hat{\Psi}_s(\vec{x}, 0)\exp(\frac{-i\hat{H}_0 t}{\hbar}),\ \hat{\Psi}^\dagger_s(\vec{x}, t) = \exp(\frac{i\hat{H}_0 t}{\hbar})\hat{\Psi}^\dagger_s(\vec{x}, 0)\exp(\frac{-i\hat{H}_0 t}{\hbar}),$$

The fields satisfy the free Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}_s(\vec{x}, t) = -\frac{\hbar^2\nabla^2}{2m}\hat{\Psi}_s(\vec{x}, t),$$

and we use this to determine $G^{(0)}_{s,s'}(\vec{x}, t; \vec{x'}, t')$. We start by differentiating the operator product that appears in $G^{(0)}_{s,s'}(\vec{x}, t; \vec{x'}, t')$,

$$i\hbar\frac{\partial}{\partial t}T[\hat{\Psi}_s(\vec{x}, t)\hat{\Psi}^\dagger_{s'}(\vec{x'}, t')]$$

$$= i\hbar\frac{\partial}{\partial t}\left\{\hat{\Psi}_s(\vec{x}, t)\hat{\Psi}^\dagger_{s'}(\vec{x'}, t')\theta(t - t')\ - \hat{\Psi}^\dagger_{s'}(\vec{x'}, t')\hat{\Psi}_s(\vec{x}, t)\theta(t' - t)\right\}$$

$$= i\hbar\left\{\hat{\Psi}_s(\vec{x}, t)\hat{\Psi}^\dagger_{s'}(\vec{x'}, t') + \hat{\Psi}^\dagger_{s'}(\vec{x'}, t')\hat{\Psi}_s(\vec{x}, t)\right\}\delta(t - t')$$

$$-\frac{\hbar^2\nabla^2}{2m}\left\{\hat{\Psi}_s(\vec{x}, t)\hat{\Psi}^\dagger_{s'}(\vec{x'}, t')\theta(t - t')\ - \hat{\Psi}^\dagger_{s'}(\vec{x'}, t')\hat{\Psi}_s(\vec{x}, t)\theta(t' - t)\right\}$$

$$= i\hbar\delta(t-t')\delta^3(\vec{x}-\vec{x'})\delta_{s,s'} - \frac{\hbar^2\nabla^2}{2m}T[\hat{\Psi}_s(\vec{x},t)\hat{\Psi}^\dagger_{s'}(\vec{x'},t')],$$

where we made use of the equal time anti-commutator of $\hat{\Psi}$ with $\hat{\Psi}^\dagger$, and the fact that spacial derivatives can be pulled out of the T symbol. (We also used $\partial_t\theta(t-t') = \delta(t-t')$, and $\partial_t\theta(t'-t) = -\delta(t-t')$.)

Returning to $G^{(0)}_{s,s'}(\vec{x},t;\vec{x'},t')$ by taking the matrix element, we now have the differential equation,

$$(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\nabla^2}{2m})G^{(0)}_{s,s'}(\vec{x},t;\vec{x'},t') = \delta_{s,s'}\delta(t-t')\delta^3(\vec{x}-\vec{x'}). \tag{2}$$

Like most cases involving Green functions, this equation is best solved using Fourier transforms. We introduce the Fourier representation of $G^{(0)}_{s,s'}(\vec{x},t;\vec{x'},t')$,

$$G^{(0)}_{s,s'}(\vec{x},t;\vec{x'},t') = \frac{1}{V}\sum_{\vec{k}}\int\frac{d\omega}{2\pi}\exp(-i\omega(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x'}))\tilde{G}^{(0)}(\omega,\vec{k})$$

Using the fact that the Fourier transform of $\delta(t-t')\delta^3(\vec{x}-\vec{x'})$ is unity, Eq.(2) becomes

$$\frac{1}{V}\sum_{\vec{k}}\int\frac{d\omega}{2\pi}\exp(-i\omega(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x'}))\left(\hbar(\omega-\omega_{\vec{k}})\tilde{G}^{(0)}(\omega,\vec{k}) - \delta_{s,s'}\right) = 0, \tag{3}$$

where $\hbar\omega_{\vec{k}} = (\hbar\vec{k})^2/2m$. Eq.(3) implies

$$\hbar(\omega-\omega_{\vec{k}})\tilde{G}^{(0)}(\omega,\vec{k}) = \delta_{s,s'}.$$

We cannot divide through by $\hbar(\omega-\omega_{\vec{k}})$ to obtain $\tilde{G}^{(0)}(\omega,\vec{k})$ until we have a prescription for what to do when $\omega = \omega_{\vec{k}}$. To see how to resolve this, look back at Eq.(1), and consider what can happen in the two possible time orders. For $t > t'$, the operator $\hat{\Psi}^\dagger_{s'}(\vec{x'},t')$ acts directly on $|\Phi^{(0)}_N>$, followed by $\hat{\Psi}_s(\vec{x},t)$, at which point the system must be back in $<\Phi^{(0)}_N|$. Now $\hat{\Psi}^\dagger_{s'}(\vec{x'},t')$ contains only creation operators, $a^\dagger_{\vec{k},s}$. Such a creation operator acting on an already filled state must give zero (Pauli principle). So the action of $\hat{\Psi}^\dagger_{s'}(\vec{x'},t')$ on $|\Phi^{(0)}_N>$ vanishes, unless $|\vec{k}| > k_F$. Once such a "particle" has been created, to get back to $<\Phi^{(0)}_N|$, the operator $\hat{\Psi}_s(\vec{x},t)$, must destroy it. So for $t > t'$, we have

$$G^{(0)}_{s,s'}(\vec{x},t;\vec{x'},t') = \frac{-i}{\hbar}\frac{1}{V}\sum_{\vec{k}}\exp(-i\omega_{\vec{k}}(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x'}))\theta(|\vec{k}|-k_F)\theta(t-t')\delta_{s,s'}$$

For the opposite time order, $t' > t$, the operator $\hat{\Psi}_s(\vec{x},t)$ acts directly on $|\Phi^{(0)}_N>$, followed by $\hat{\Psi}^\dagger_{s'}(\vec{x'},t')$. Now $\hat{\Psi}_s(\vec{x},t)$ contains only destruction operators, $a_{\vec{k},s}$. These can only give a non-zero result if the state is occupied, or we must have $|\vec{k}| < k_F$. This is often termed creating a "hole." To get to $<\Phi^{(0)}_N|$, the operator $\hat{\Psi}^\dagger_{s'}(\vec{x'},t')$ must then create the particle just destroyed or in hole language, destroy the hole. With this information, we can write a formula for the partial Fourier transform of $G^{(0)}$, going from $\vec{x}$ space to $\vec{k}$ space. We have

$$\tilde{G}^{(0)}_{s,s'}(t-t',\vec{k}) = \frac{-i}{\hbar}\exp(-i\omega_{\vec{k}}(t-t'))\left(\theta(|\vec{k}|-k_F)\theta(t-t') - \theta(|k_F-|\vec{k}|)\theta(t'-t)\right)\delta_{s,s'}$$

Multiplying by $\exp(i\omega(t-t'))$ and integrating on time, we finally have

$$\tilde{G}^{(0)}_{s,s'}(\omega,\vec{k}) = \frac{\delta_{s,s'}}{\hbar}\left(\frac{\theta(|\vec{k}|-k_F)}{\omega-\omega_{\vec{k}}+i\epsilon} + \frac{\theta(|k_F-|\vec{k}|)}{\omega-\omega_{\vec{k}}-i\epsilon}\right). \tag{4}$$

The signs of the $i\epsilon$ terms in Eq.(4) arise from the damping needed in the time integrals at $\pm\infty$. We may unify this into one expression,

$$\tilde{G}^{(0)}_{s,s'}(\omega,\vec{k}) = \frac{\delta_{s,s'}}{\hbar}\left(\frac{1}{\omega-\omega_{\vec{k}}+i\epsilon(k-k_F}\right),$$

where

$$\epsilon(k-k_F) = \begin{cases} +\epsilon & k>k_F \\ -\epsilon & k<k_F \end{cases}$$

The poles of $\tilde{G}^{(0)}_{s,s'}(\omega,\vec{k})$ in ω are still at $\omega_{\vec{k}}$, so this is still a free particle. The prescription for how to handle the behavior at the pole is the only indication of the presence of a Fermi sea. This changes when we go to the interacting Green function.

Exact Propagator and Perturbation Theory Rewriting the full equation of motion for the field $\hat{\Psi}$, we have

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}_s(\vec{x},t) = -\frac{\hbar^2\nabla^2}{2m}\hat{\Psi}_s(\vec{x},t) + \sum_{\bar{s}}\int d\vec{y}V(\vec{y}-\vec{x})\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_{\bar{s}}(\vec{y},t)\hat{\Psi}_s(\vec{x},t).$$

The exact propagator is defined by

$$G_{s,s'}(\vec{x},t;\vec{x'},t') = -\frac{i}{\hbar} < \Psi^{(0)}_N|T[\hat{\Psi}_s(\vec{x},t)\hat{\Psi}^\dagger_{s'}(\vec{x'},t')]|\Psi^{(0)}_N >,$$

where $|\Psi^{(0)}_N >$ is the exact ground state, and $\hat{\Psi}$ and $\hat{\Psi}^\dagger$ move in time with the full Hamiltonian. Using the equation of motion for $\hat{\Psi}$, we have the following equation for the exact propagator,

$$(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\nabla^2}{2m})G_{s,s'}(\vec{x},t;\vec{x'},t') = \delta_{s,s'}\delta(t-t')\delta^3(\vec{x}-\vec{x'}) \tag{5}$$

$$-\frac{i}{\hbar}\sum_{\bar{s}}\int d\vec{y}V(\vec{y}-\vec{x}) < \Psi^{(0)}_N|T[\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_{\bar{s}}(\vec{y},t)\hat{\Psi}_s(\vec{x},t)\hat{\Psi}^\dagger_{s'}(\vec{x'},t')]|\Psi^{(0)}_N >$$

In Eq.(5), the δ function term is the same as for the free propogator because the anti commuation relations at equal times are independent of how the operators move in time.

For a normal Fermi system, Eq.(5) can be usefully studied in perturbation theory. The potential is manifestly of first order, so the equation for the first order propagator can be found by setting all other quantities on the right side of the equation equal to their zeroth order values. This gives,

$$(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2\nabla^2}{2m})G^{(1)}_{s,s'}(\vec{x},t;\vec{x'},t')$$
$$=-\frac{i}{\hbar}\sum_{\bar{s}}\int d\vec{y}V(\vec{y}-\vec{x})<\Phi_N^{(0)}|T[\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_{\bar{s}}(\vec{y},t)\hat{\Psi}_s(\vec{x},t)\hat{\Psi}^\dagger_{s'}(\vec{x'},t')]|\Phi_N^{(0)}>$$

The key to making progress in solving this equation is understanding the matrix element which multiplies $V(\vec{y}-\vec{x})$. Writing this out, we have

$$-\frac{i}{\hbar}<\Phi_N^{(0)}|T[\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_{\bar{s}}(\vec{y},t)\hat{\Psi}_s(\vec{x},t)\hat{\Psi}^\dagger_{s'}(\vec{x'},t')]|\Phi_N^{(0)}>$$
$$=-\frac{i}{\hbar}<\Phi_N^{(0)}|\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_{\bar{s}}(\vec{y},t)\hat{\Psi}_s(\vec{x},t)\hat{\Psi}^\dagger_{s'}(\vec{x'},t')|\Phi_N^{(0)}>\theta(t-t')$$
$$+\frac{i}{\hbar}<\Phi_N^{(0)}|\hat{\Psi}^\dagger_{s'}(\vec{x'},t')\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_{\bar{s}}(\vec{y},t)\hat{\Psi}_s(\vec{x},t)|\Phi_N^{(0)}>\theta(t'-t)$$

We will analyze the term with $t>t'$ in detail. The other time order is handled with similar methods. We note that for $t>t'$, the operator $\hat{\Psi}^\dagger_{s'}(\vec{x'},t')$ acts directly on $|\Phi_N^{(0)}>$. Since $\hat{\Psi}^\dagger_{s'}(\vec{x'},t')$ contains only creation operators, it must give a vanishing result on any occupied state with $k<k_F$. So it must create a particle with $k>k_F$. Since we are ultimately taking a matrix element in $|\Phi_N^{(0)}>$, either $\hat{\Psi}_s(\vec{x},t)$ or $\hat{\Psi}_{\bar{s}}(\vec{y},t)$ must destroy this particle. This gives the two terms written out in the following equation,

$$-\frac{i}{\hbar}<\Phi_N^{(0)}|\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_{\bar{s}}(\vec{y},t)\hat{\Psi}_s(\vec{x},t)\hat{\Psi}^\dagger_{s'}(\vec{x'},t')|\Phi_N^{(0)}>\theta(t-t') \tag{6}$$
$$=<\Phi_N^{(0)}|\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_{\bar{s}}(\vec{y},t)|\Phi_N^{(0)}>\left(-\frac{i}{\hbar}<\Phi_N^{(0)}|\hat{\Psi}_s(\vec{x},t)\hat{\Psi}^\dagger_{s'}(\vec{x'},t')|\Phi_N^{(0)}>\right)\theta(t-t')$$
$$-<\Phi_N^{(0)}|\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_s(\vec{x},t)|\Phi_N^{(0)}>\left(-\frac{i}{\hbar}<\Phi_N^{(0)}|\hat{\Psi}_{\bar{s}}(\vec{y},t)\hat{\Psi}^\dagger_{s'}(\vec{x'},t')|\Phi_N^{(0)}>\right)\theta(t-t').$$

In writing Eq.(6) we have used the fact that once the particle created by $\hat{\Psi}^\dagger_{s'}(\vec{x'},t')$ is destroyed, the system is back in $|\Phi_N^{(0)}>$. Note also the second term in Eq.(6) has a minus sign because two Fermi operators have been moved past each other. The factors multiplying $\theta(t-t')$ amount to a factor of the free propagator, so we have (still only for $t>t'$,)

$$(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2\nabla^2}{2m})G^{(1)}_{s,s'}(\vec{x},t;\vec{x'},t')$$
$$=\sum_{\bar{s}}\int d\vec{y}V(\vec{y}-\vec{x})\Big\{<\Phi_N^{(0)}|\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_{\bar{s}}(\vec{y},t)|\Phi_N^{(0)}>G^{(0)}_{s,s'}(\vec{x},t;\vec{x'},t')$$
$$-<\Phi_N^{(0)}|\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_s(\vec{x},t)|\Phi_N^{(0)}>G^{(0)}_{\bar{s},s'}(\vec{y},t;\vec{x'},t')\Big\}$$

The first term is very simple. The quantity $\sum_{\bar{s}}\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_{\bar{s}}(\vec{y},t)$ is just the density of particles, so this term gives

$$\sum_{\bar{s}}<\Phi_N^{(0)}|\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_{\bar{s}}(\vec{y},t)|\Phi_N^{(0)}>=\frac{N}{V}.$$

For the second term, we note that $\hat{\Psi}_s(\vec{x},t)$ can only destroy a particle with $k < k_F$, and to get back to $|\Phi_N^{(0)}>$, $\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)$ must create it again. This gives

$$< \Phi_N^{(0)}|\hat{\Psi}^\dagger_{\bar{s}}(\vec{y},t)\hat{\Psi}_s(\vec{x},t)|\Phi_N^{(0)} >= \delta_{\bar{s},s}\left(\sum_{k'<k_F}\frac{1}{V}\exp(i\vec{k'}\cdot(\vec{x}-\vec{y}))\right)$$

Our differential equation for the first order propagator now reads,

$$(i\hbar\frac{\partial}{\partial t}+\frac{\hbar^2\nabla^2}{2m})G^{(1)}_{s,s'}(\vec{x},t;\vec{x'},t') = \tag{7}$$

$$\int d\vec{y}V(\vec{y}-\vec{x})\left(\frac{N}{V}G^{(0)}_{s,s'}(\vec{x},t;\vec{x'},t')\ -(\sum_{k'<k_F}\frac{1}{V}\exp(i\vec{k'}\cdot(\vec{x}-\vec{y})))G^{(0)}_{s,s'}(\vec{y},t;\vec{x'},t')\right).$$

The next step is to Fourier analyze. We do this in stages, first Fourier transforming in space, then in time. Introducing the Fourier transform of the potential, we have

$$V(\vec{y}-\vec{x}) = \frac{1}{V}\sum_{\vec{q}}\tilde{V}(\vec{q})\exp(i\vec{q}(\cdot\vec{y}-\vec{x})).$$

Now tackling the first term on the right side of Eq.(7) we have

$$\int d\vec{y}V(\vec{y}-\vec{x}) = \frac{1}{V}\sum_{\vec{q}}\tilde{V}(\vec{q})V\delta_{\vec{q},0}\exp(-i\vec{q}\cdot\vec{x}),=\tilde{V}(0)$$

so this term becomes

$$\frac{N}{V}\tilde{V}(0)G^{(0)}_{s,s'}(\vec{x},t;\vec{x'},t') = \frac{N}{V}\tilde{V}(0)\frac{1}{V}\sum_{\vec{k}}\tilde{G}^{(0)}_{s,s'}(t-t',\vec{k})\exp(i\vec{k}\cdot(\vec{x}-\vec{x'})),$$

where we introduced the spacial Fourier transform of the free propagator. The second term on the right side of Eq.(7) is a little more intricate. We have

$$-\int d\vec{y}\frac{1}{V}\sum_{\vec{q}}\tilde{V}(\vec{q})\exp(i\vec{q}\cdot(\vec{y}-\vec{x}))\frac{1}{V}\sum_{k'<k_F}\exp(i\vec{k'}\cdot(\vec{x}-\vec{y}))\frac{1}{V}\sum_{\vec{k}}\tilde{G}^{(0)}_{s,s'}(t-t',\vec{k})\exp(i\vec{k}\cdot(\vec{y}-\vec{x'}))$$

Doing the integral on $\vec{y}$ gives a factor $V\delta_{\vec{q}-\vec{k'}+\vec{k}}$. This can be used to eliminate the sum over $\vec{q}$, and we finally have

$$= -\frac{1}{V}\sum_{\vec{k}}\frac{1}{V}\sum_{k'<k_F}\tilde{V}(\vec{k'}-\vec{k})\tilde{G}^{(0)}_{s,s'}(t-t',\vec{k})\exp(i\vec{k}\cdot(\vec{x}-\vec{x'})).$$

Finally, we Fourier transform in time, multiplying by $\exp(i\omega(t-t'))$ and integrating on time. We obtain,

$$\hbar(\omega-\omega_{\vec{k}})\tilde{G}^{(1)}_{s,s'}(\omega,\vec{k}) = \left(\frac{N}{V}\tilde{V}(0)-(\frac{1}{V}\sum_{|\vec{k'}|<k_F}\tilde{V}(\vec{k'}-\vec{k}))\right)\tilde{G}^{(0)}_{s,s'}(\omega,\vec{k})$$

Figure 1: Feynman Diagrams for First Order Propagator

Solving for $\tilde{G}^{(1)}_{s,s'}(\omega, \vec{k})$ by dividing by $\hbar(\omega - \omega_{\vec{k}})$, we have

$$\tilde{G}^{(1)}_{s,s'}(\omega, \vec{k}) = \sum_{\bar{s}} \tilde{G}^{(0)}_{s,\bar{s}}(\omega, \vec{k}) \left(\frac{N}{V}\tilde{V}(0) - (\frac{1}{V} \sum_{|\vec{k'}|<k_F} \tilde{V}(\vec{k'} - \vec{k})) \right) \tilde{G}^{(0)}_{\bar{s},s'}(\omega, \vec{k}), \tag{8}$$

where we have inserted a dummy sum over $\bar{s}$, since this will occur in higher order perturbation theory. The two terms are shown as Feynman diagrams in the figure. The quantity sandwiched between the two free propagators in Eq.(8) is a *self energy.* What we have done in the preceding steps is calculate the self energy to first order. The standard letter for self energies is Σ, and we have Fourier transformed, so our formula is

$$\tilde{\Sigma}^{(1)}(\omega, \vec{k}) = \left(\frac{N}{V}\tilde{V}(0) - (\frac{1}{V} \sum_{k'<k_F} \tilde{V}(\vec{k'} - \vec{k})) \right)$$

In first order the self-energy is independent of ω, but dependence on ω appears in higher orders.

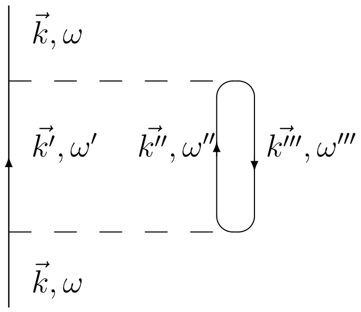
The Self Energy and Summing Perturbation Theory If we go on to higher order perturbation theory, we will find a series of the form

$$\tilde{G} = \tilde{G}^{(0)} + \tilde{G}^{(0)}\tilde{\Sigma}\,\tilde{G}^{(0)} + \tilde{G}^{(0)}\tilde{\Sigma}\,\tilde{G}^{(0)}\tilde{\Sigma}\,\tilde{G}^{(0)} + \dots$$

where all spin indices have been suppressed and the arguments $\omega, \vec{k}$ have not been written. The self energy itself has a perturbation expansion,

$$\tilde{\Sigma} = \tilde{\Sigma}^{(1)} + \tilde{\Sigma}^{(2)} + \dots$$

Figure 2: Second Order term in the Propagator



$$\vec{k'''} = \vec{k''} + \vec{k'} - \vec{k}$$
$$\omega''' = \omega' + \omega'' - \omega$$

Technical Note What is being discussed here is the so-called *irreducible* self energy, meaning only terms which cannot be divided in two parts by cutting one fermion line. Formally summing the series, we have

$$\tilde{G} = \tilde{G}^{(0)} \frac{1}{1 - \tilde{\Sigma}\tilde{G}^{(0)}} = \frac{1}{(\tilde{G}^{(0)})^{-1} - \tilde{\Sigma}}$$

The quantity $(\tilde{G}^{(0)})^{-1}$ is just $\hbar(\omega - \omega_{\vec{k}})$. The zeroes of the denominator of $\tilde{G}$ are the *quasi-particles* of the system, and are found by setting

$$\hbar(\omega - \omega_{\vec{k}}) - \tilde{\Sigma}(\omega, \vec{k}) = 0.$$

The poles of $\tilde{G}_{s,s'}$ will not be on the real *omega* axis, signifying that the excitations have finite lifetimes. However, when $|\vec{k}$ is very close to $k_F <$ the poles move to the real axis and the corresponding lieftimes become long. These so-called "quasi-particles" control much of the important physics of the system.

Summary on taking space-time to wave vector-frequency

- A factor $\exp(i\vec{k} \cdot (\vec{x} - \vec{y}))$ represents wave vector $\vec{k}$ flowing *from $\vec{y}$ to $\vec{x}$.*
- A factor $\exp(-i\omega(t - t'))$ represents frequency ω flowing *from* time t' *to* time t.
- There is a spacial integral at each *vertex* of the graph. The result is that the sum of wave vectors flowing into the vertex equals the sum flowing out of the vertex. This represents conservation of momentum.

- There is an integral over all times *internal* to the graph. This results in δ functions which represent conservation of energy. (The frequency rule is applied to *pairs* of vertices connected by potential lines. The total frequency into such a pair equals the total frequency out. This difference compared to the wave vector case is caused by the fact that the potential is static (instantaneous).

- To draw the Feynman graphs for a given contribution, a Green function is a solid line, and a potential is a wavy or dotted line. The direction of wave vectors is represented by an arrow near the line in the drawing.

- Green functions are matrices in spin indices. If the potential has no spin dependence, this matrix structure is trivial (all $G^{(0)}$ identity matrices.)

- Once the diagram is completely in wave vector-frequency space, there will remain sums over wave vectors and integrals over frequencies in closed loops.

Expression for the graph of Fig.(4) The contribution to the self energy of the graph in Fig.(4) is the following:

$$\hbar^2(\frac{1}{V}\sum_{k'}\int\frac{d\omega'}{2\pi})(\frac{1}{V}\sum_{k''}\int\frac{d\omega''}{2\pi})G^{(0)}_{s,s'}(\omega',\vec{k'})\tilde{V}(\vec{k}-\vec{k'})\tilde{V}(\vec{k}-\vec{k'})\cdot \tag{9}$$

$$\cdot\left(\sum_{s'',\bar{s}}G^{(0)}_{\bar{s},s''}(\omega'',\vec{k''})G^{(0)}_{s'',\bar{s}}(\omega'+\omega''-\omega,\vec{k'}+\vec{k''}-\vec{k})\right)$$

To derive this expression, one can start from the space time version of Fig.(4), then substitute the Fourier expansion for every Green function line and potential in the graph. After doing the integrals over all internal space time coordinates, one arrives at an expression which has the two external $G^{(0)}$ lines attached, with the expression above sandwiched in between (along with other contributions to the second order self energy).

To derive the space time expression itself, one considers the following expression for the exact Green function,

$$(\frac{-i}{\hbar}) < \Psi^{(0)}_N|T\left(\Psi_s(\vec{x},t)U_I(\infty,-\infty)\Psi^\dagger_{s'}(\vec{x'},t')\right)|\Psi^{(0)}_N>_{con} \tag{10}$$

Here the $U_I(\infty,-\infty)$ operator is

$$U_I(\infty,-\infty) = I - \frac{i}{\hbar}\int_{-\infty}^{+\infty}dt''\hat{V}_I(t'') + (\frac{-i}{\hbar})^2\int_{-\infty}^{\infty}dt''\int_{-\infty}^{t''}dt'''\hat{V}_I(t'')\hat{V}_I(t''') + \dots, \tag{11}$$

where $\hat{V}$ is the formula for the interaction for a Fermi system as in Eq.(6) of the previous section. The subscript *con* in Eq.(10)means "connected" so that graphs with no external $G^{(0)}$ lines are omitted. In the derivation of Eq.(10) given in books on many body quantum mechanics, (e.g. Fetter and Walecka, *Quantum Theory of Many Particle Systems*, it is shown that the evaluation of the T product or time-ordering is to be done *prior* to the integrals over t'', t'''. The technical tool used to go from expressions involving products of several field operators to space time graphs is called Wick's theorem.