Photons

In this section, we want to quantize the vector potential. We will work in the *Coulomb* gauge, where

$$\vec{\nabla} \cdot \vec{A} = 0.$$

In gaussian units, the fields are defined as follows,

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi,$$

$$\vec{B} = \vec{\nabla} \times \vec{A},$$

where $\vec{A}$ and ϕ are the vector and scalar potentials. In the Coulomb gauge, the vector and scalar potentials are decoupled. This can be seen by looking at Gauss's Law. We have

$$\vec{\nabla} \cdot \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{A} - \nabla^2\phi = -\nabla^2\phi = 4\pi\rho,$$

where ρ is the charge density. The last equality means we can solve for ϕ in the usual way using Coulomb's Law:

$$\phi(\vec{x}, t) = \int d\vec{x'}\,\frac{\rho(\vec{x'}, t)}{|\vec{x} - \vec{x'}|}$$

This has the consequence that the Coulombic part of the electric field is handled as always. We simply write down the terms in the Hamiltonian coming from the Coulomb interactions between the various charges present, electrons and nuclei for an atom, molecule or condensed matter system. The vector potential $\vec{A}$ then describes photons.

Lagrangian for Photons We will use the symbol γ to denote quantities having to do with photons. Following what was done for the string, we need to write down the Lagrangian for photons (non-interacting, free photons.) The part of the electric field coming from photons is

$$-\frac{1}{c}\frac{\partial \vec{A}}{\partial t}.$$

This is analogous to

$$\frac{\partial u}{\partial t}$$

for the string. The $\vec{B}$ field is

$$\vec{\nabla} \times \vec{A},$$

analogous to

$$\frac{\partial u}{\partial x}$$

for the string. Imitating the structure of the string Lagrangian, we can deduce the form of the Lagrangian for photons. We have

$$L^{\gamma} = \frac{1}{8\pi}\int d\vec{x}\left((-\frac{1}{c}\frac{\partial \vec{A}}{\partial t})^2 - (\vec{\nabla}\times\vec{A})^2\right)$$

The factor $1/8\pi$ in front of the integral can be traced back to the factors of 4π that appear in Maxwell's equations in gaussian units:

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

Equation of Motion for Free Photons As with any system, the equation of motion is derived from the action principle. We will imbed our system in a spacial box with periodic boundary conditions and integrate over a time interval $0, T$. This gives

$$S = \int_0^T dt\frac{1}{8\pi}\int_V d\vec{x}\left((-\frac{1}{c}\frac{\partial \vec{A}}{\partial t})^2 - (\vec{\nabla}\times\vec{A})^2\right).$$

We now vary $\vec{A}$,

$$\vec{A} \to \vec{A} + \delta\vec{A}$$

with

$$\delta\vec{A}(\vec{x},0) = \delta\vec{A}(\vec{x},T) = 0,$$

$$\delta\vec{A}(0,y,z,t) = \delta\vec{A}(L_x,y,z,t),\ etc$$

Substituting and keeping only linear terms in $\delta\vec{A}$, we obtain

$$\delta S = \frac{1}{4\pi}\int_0^T dt\int_V d\vec{x}\left(\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\cdot\frac{1}{c}\frac{\delta\vec{A}}{\partial t} - (\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\cdot\delta\vec{A})\right)$$

The first term in δS is straightforward to integrate by parts, and gives

$$\frac{1}{4\pi}\int_0^T dt\int_V d\vec{x}\,\delta\vec{A}\cdot\left(\frac{-1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}\right).$$

The second term needs a little rearrangement. Using the identity

$$\vec{U}\cdot(\vec{V}\times\vec{W}) = \vec{W}\cdot(\vec{U}\times\vec{V}) = -\vec{W}\cdot(\vec{V}\times\vec{U})$$

we have

$$-(\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\delta\vec{A}) = -\delta\overset{\downarrow}{\vec{A}}\cdot\left((\vec{\nabla}\times\vec{A})\times\overset{\uparrow}{\vec{\nabla}}\right) = \delta\overset{\downarrow}{\vec{A}}\cdot\left(\overset{\uparrow}{\vec{\nabla}}\times(\vec{\nabla}\times\vec{A}\right),$$

where the ↑ and ↓ remind us which $\vec{\nabla}$ acts on $\delta\vec{A}$. An integration by parts will transfer all $\vec{\nabla}$ so that they act on $\vec{A}$. We finally get for δS,

$$\delta S = \frac{1}{4\pi}\int_0^T dt \int_V d\vec{x}\, \delta\vec{A}\cdot\left(\frac{-1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \vec{\nabla}\times(\vec{\nabla}\times\vec{A})\right)$$

Using

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = -\nabla^2\vec{A},$$

we have

$$\delta S = \frac{1}{4\pi}\int_0^T dt \int_V d\vec{x}\, \delta\vec{A}\cdot\left(\frac{-1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} + \nabla^2\vec{A}\right)$$

Demanding that $\delta S = 0$ for arbitrary $\delta\vec{A}$ finally gives the wave equation,

$$\frac{-1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} + \nabla^2\vec{A} = 0$$

Expansion of $\vec{A}$ in Creation and Destruction Operators We can deduce the correct expansion of $\vec{A}$ in terms of photon creation and destruction operators by comparing to the case of the string for periodic boundary conditions. For the string we had

$$L = \int_0^B dx\left(\frac{\rho}{2}(\frac{\partial u}{\partial t})^2 - \frac{T}{2}(\frac{\partial u}{\partial x})^2\right)$$

for the Lagrangian and

$$u(x,t) = \sum_k{}' \sqrt{\frac{\hbar}{2B\rho\omega_k}}\left(a_k e^{ikx-i\omega_k t} + a_k^\dagger e^{-ikx+i\omega_k t}\right)$$

for the expansion in creation and destruction operators for periodic boundary conditions. Going from a one dimensional string with periodic boundary conditions to a three dimensional box with periodic boundary conditions is simple. We have

$$\frac{1}{\sqrt{B}}\exp(ikx) \rightarrow \frac{1}{\sqrt{V}}\exp(i\vec{k}\cdot\vec{x}),$$

where

$$\vec{k} = (\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y}, \frac{2\pi n_z}{L_z}),$$

and $V = L_x L_y L_z$. To obtain the correct square root factor, we compare the terms in time derivatives in the two Lagrangians. For the string, we have

$$\int_0^B dx\left(\frac{\rho}{2}(\frac{\partial u}{\partial t})^2\right),$$

while for photons we have

$$\int d\vec{x} \left(\frac{1}{8\pi c^2} (\frac{\partial \vec{A}}{\partial t})^2 \right).$$

We see that in going from strings to photons,

$$\rho \to \frac{1}{4\pi c^2}.$$

We now can make an educated (and correct) guess that the expansion of $\vec{A}$ is

$$\vec{A}(\vec{x}) = \sum_{\vec{k}} \sqrt{\frac{4\pi\hbar c^2}{2\omega_{\vec{k}}}} \left(\vec{a}_{\vec{k}} \frac{1}{\sqrt{V}} \exp(i\vec{k} \cdot \vec{x}) + \vec{a}^{\dagger}_{\vec{k}} \frac{1}{\sqrt{V}} \exp(-i\vec{k} \cdot \vec{x}) \right),$$

where from the wave equation, $\omega^2_{\vec{k}} = k^2 c^2$. To insure that $\vec{\nabla} \cdot \vec{A} = 0$, we must have

$$\vec{k} \cdot \vec{a}_{\vec{k}} = 0.$$

This is done by using *polarization* vectors. We expand

$$\vec{a}_{\vec{k}} = \sum_{\lambda} a_{\vec{k}\lambda} \vec{\epsilon}_{\lambda}(\vec{k}),$$

where the $\vec{\epsilon}_{\lambda}(\vec{k})$ are any two linearly independent vectors satisfying

$$\vec{\epsilon}_{\lambda}(\vec{k}) \cdot \vec{k} = 0.$$

A common and useful choice is to use so-called circular polarization vectors. For $\vec{k}$ along the $+z$ axis, we would have

$$\vec{\epsilon}_1(\vec{k}) = \frac{1}{\sqrt{2}}(\hat{e}_1 + i\hat{e}_2), \quad \vec{\epsilon}_{-1}(\vec{k}) = \frac{1}{\sqrt{2}}(\hat{e}_{-1} - i\hat{e}_2)$$

For an arbitrary direction of $\vec{k}$, we rotate the system composed of the z axis, ϵ_1 and ϵ_2 to the direction of $\vec{k}$.

The expansion of $\vec{A}$ can now be written,

$$\vec{A}(\vec{x}) = \sum_{\vec{k}\lambda} \sqrt{\frac{4\pi\hbar c^2}{2\omega_{\vec{k}} V}} \left(a_{\vec{k}\lambda} \vec{\epsilon}_{\lambda}(\vec{k}) \exp(i\vec{k} \cdot \vec{x}) + a^{\dagger}_{\vec{k}\lambda} \vec{\epsilon}^{*}_{\lambda}(\vec{k}) \exp(i\vec{k} \cdot \vec{x}) \right), \tag{1}$$

To gain insight into the indices on the polarization vectors, consider a photon traveling up the $z$axis with polarization vector $\vec{\epsilon}_1$. Denoting the "no photon" state by $|0>$, i.e. all the harmonic oscillators in their ground states, this one photon state would be

$$a^{\dagger}_{k\hat{z},1}|0> .$$

Calculating the matrix element $< 0|\vec{A}(\vec{x})a^\dagger_{k\hat{z},1}|0>$ we have

$$< 0|\vec{A}(\vec{x})a^\dagger_{k\hat{z},1}|0 >= \sqrt{\frac{4\pi\hbar c^2}{2\omega_{\vec{k}}V}}\exp(ikz)\frac{1}{\sqrt{2}}(\hat{e}_1 + i\hat{e}_2) \tag{2}$$

(This calculation is easy. Just substitute the expansion for $\vec{A}$ and use the commutation rules

$$[a_{\vec{k}\lambda}, a^\dagger_{\vec{k}'\lambda'}] = \delta_{\vec{k},\vec{k}'}\delta_{\lambda,\lambda'})$$

Now looking at Eq.(2) we see a plane wave headed up the z axis, multiplied by $\hat{e}_1 + i\hat{e}_2$. This expression is basically the wave function of the photon, with the polarization vector being the spin wave function. The $1 + i2$ character of $\vec{\epsilon}_1$ implies that the spin projection along $\vec{k}$ is $+1$, so for a general direction of $\vec{k}$, $\vec{\epsilon}_{\pm 1}(\vec{k})$ corresponds to a photon spin projection along $\vec{k}$ of ± 1 (in units of $\hbar$.) This clarifies the meaning of the indices on polarization vectors and the meaning of the statement that "a photon has spin 1." This is true, but it is also important to note that the spin projection of a photon along its wave vector can only be ± 1, never 0.

Time Dependence of the Vector Potential The Hamiltonian of the system of free photons is straightforward to write down. In terms of creation and destruction operators, we have

$$H_\gamma = \sum_{\vec{k},\lambda} \hbar\omega_{\vec{k}}(a^\dagger_{\vec{k},\lambda}a_{\vec{k},\lambda} + \frac{1}{2}), \tag{3}$$

the expected form for a set of harmonic oscillators. It is also of interest to write H_γ in terms of electric and magnetic fields. Recall the formula for the electric field;

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi.$$

The Coulomb gauge allows us to cleanly split $\vec{E}$ into a photon part and a Coulomb part. We have

$$\vec{E} = \vec{E}_\gamma + \vec{E}_{coul},$$

where

$$\vec{E}_\gamma = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \text{ and } \vec{E}_{coul} = -\vec{\nabla}\phi,$$

and since we are not presently considering applied external magnetic fields, we also have

$$\vec{B} = \vec{B}_\gamma = \vec{\nabla}\times\vec{A}.$$

In terms of $\vec{E}_\gamma$ and $\vec{B}_\gamma$, the Hamiltonian is also given by

$$H_\gamma = \frac{1}{8\pi}\int_V d\vec{x}\left((\vec{E}_\gamma)^2 + (\vec{B}_\gamma)^2\right) \tag{4}$$

Showing that Eq.(3) is the same as Eq.(4) is done by using the expansion of $\vec{A}$ from Eq.(1) to calculate the electric and magnetic fields, and carrying out the integral $d\vec{x}$ in Eq.(4). Both Eqs.(3) and (4) represent the energy present in the form of photons. The rest of the electromagnetic energy is in the Coulomb interactions between charges.

It is now easy to get the time dependence of $\vec{A}$. By definition,

$$\vec{A}(\vec{x},t) = \exp(\frac{iH_\gamma t}{\hbar})\vec{A}(\vec{x},0)\exp(\frac{-iH_\gamma t}{\hbar})$$

Now since what we have is a set of independent harmonic oscillators, the time dependence of the creation and destruction operators is the usual form,

$$\exp(\frac{iH_\gamma t}{\hbar})a_{\vec{k},\lambda}\exp(\frac{-iH_\gamma t}{\hbar}) = a_{\vec{k},\lambda}\exp(-i\omega_{\vec{k}}t),$$

and

$$\exp(\frac{iH_\gamma t}{\hbar})a^\dagger_{\vec{k},\lambda}\exp(\frac{-iH_\gamma t}{\hbar}) = a^\dagger_{\vec{k},\lambda}\exp(i\omega_{\vec{k}}t),$$

Using these formulas along with Eq.(1) we finally have

$$\vec{A}(\vec{x},t) = \sum_{\vec{k}\lambda}\sqrt{\frac{4\pi\hbar c^2}{2\omega_{\vec{k}}V}}\left(a_{\vec{k}\lambda}\vec{\epsilon}_\lambda(\vec{k})\exp(i\vec{k}\cdot\vec{x} - \omega_{\vec{k}}t) + a^\dagger_{\vec{k}\lambda}\vec{\epsilon}^*_\lambda(\vec{k})\exp(i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}}t)\right), \quad (5)$$

It is worthwhile to examine the coefficients of the creation and destruction operators in Eq.(5). For the coefficient of $a_{\vec{k},\lambda}$ we have

$$\sqrt{\frac{4\pi\hbar c^2}{2\omega_{\vec{k}}}}(\frac{1}{\sqrt{V}}\exp(i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}}t)\epsilon_\lambda)$$

In calculations, this factor will accompany a photon in the *initial* state of a process. Its structure is fairly easy to understand. The factor

$$(\frac{1}{\sqrt{V}}\exp(i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}}t)\epsilon_\lambda)$$

is just a plane wave factor along with a spin wave function. In the prefactor square root

$$\sqrt{\frac{\hbar}{2\omega_{\vec{k}}}}$$

is familiar in every harmonic oscillator problem. Finally

$$\sqrt{4\pi c^2}$$

can be deduced from the $1/8\pi c^2$ accompanying

$$(\frac{\partial\vec{A}}{\partial t})^2$$

in the Lagrangian.

In a similar way, the factors accompanying $a^\dagger_{\vec{k},\lambda}$ in Eq.(5) are the complex conjugates of those accompanying $a_{\vec{k},\lambda}$, and in calculations, these will accompany a photon in the *final* state of a process.