Quantizing a String

Our real interest is learning how to describe the interaction of photons with electrons and other charges. This means quantizing the vector potential $\vec{A}(\vec{x}, t)$. This is a quantized field, i.e. a quantum operator which depends on space. To illustrate the method of quantizing fields, we start with a simpler case, that of a beaded string which we smooth out and make into a continuous string.

Discrete to Continuous String We start with a discrete system which can be transformed into a continuous system involving a field. Consider a string, held fixed at its ends. The string has a tension T, and holds a set of equal masses, m, spaced apart by a distance b. The transverse displacement of the ith mass is denoted as $u_i(t)$, and this mass is located at $x = ib$, where $i = 0, 1, \ldots, N$, and $u_0 = u_N = 0$. Our first task is to write down the Lagrangian of the system, which will be in the usual form, $L = K - V$, where K and V are kinetic and potential energies. The kinetic energy is easy to write down directly. It is just the sum of the kinetic energies of the individual masses, so we have

$$K = \sum_{i=0}^{N} \frac{m}{2} (\frac{\partial u_i}{\partial t})^2$$

To get the potential, we first consider the restoring force on the ith mass. We have for small displacements,

$$F_i = -T \left(\frac{u_i - u_{i+1}}{b} + \frac{u_i - u_{i-1}}{b} \right).$$

This force should be given by

$$F_i = -\frac{\partial V}{\partial u_i}.$$

It is easy to see that we will get the right expression if we set

$$V = \sum_{i=1}^{N} \frac{Tb}{2} (\frac{u_i - u_{i-1}}{b})^2.$$

Finally, we have the Lagrangian for the discrete system,

$$L - K - V = \frac{1}{2} \sum_i \left(m(\frac{\partial u_i}{\partial t})^2 - Tb(\frac{u_i - u_{i-1}}{b})^2 \right)$$

Now, staying with the discrete system, we introduce some notation which will make the continuum limit easier to see. We define a mass *density*

$$m = \rho b.$$

We replace the index i by the coordinate x, and the spacing b by the increment in x, Δx,

$$u_i(t) \equiv u(x, t), \quad x = ib, \quad \Delta x = b$$

Using this notation, we rewrite the Lagrangian for the discrete system,

$$L = \sum_x \Delta x \left(\frac{\rho}{2} (\frac{\partial u}{\partial t})^2 - \frac{T}{2} (\frac{\Delta u}{\Delta x})^2 \right) \tag{1}$$

It is now clear how to take the continuum limit. We let the number of masses N approach ∞, take the individual mass m to zero, along with the spacing Δx. Held fixed in this limit are $L_x = N\Delta x$, and $\rho = m/\Delta x$. The sum over x becomes an integral over x, and the Lagrangian takes the form

$$L = \int_0^{L_x} dx \left(\frac{\rho}{2} (\frac{\partial u}{\partial t})^2 - \frac{T}{2} (\frac{\partial u}{\partial x})^2 \right) \tag{2}$$

It is characteristic of a continuous system that the Lagrangian is the integral of a Lagrange *density.*

Equations of Motion Given a Lagrangian directly in continuum form like Eq.(2), it is fair to ask how the equations of motion are found. The answer is to use the action of the system, which as always is the integral of the Lagrangian over a time interval. For the continuous string, the action would be given by

$$S = \int_0^T dt \int_0^{L_x} dx \frac{1}{2} \left(\rho (\frac{\partial u}{\partial t})^2 - T (\frac{\partial u}{\partial x})^2 \right)$$

The action principle states that S is stationary to first order when

$$u \to u + \delta u,$$

if we also require fixed endpoints in space and time,

$$\delta u(0,t) = \delta u(L_x, t) = \delta u(x, 0) = \delta u(x, T) = 0$$

Varying u in this way, and integrating by parts, we have

$$\delta S = -\int_0^T dt \int_0^{L_x} dx \left(\rho (\frac{\partial^2 u}{\partial t^2}) - T (\frac{\partial^2 u}{\partial x^2}) \right) \delta u(x,t)$$

Demanding that $\delta S = 0$, for *arbitrary* $\delta u(x,t)$ then requires that the coefficient of δu in the action integral must vanish, or

$$(\frac{\partial^2 u}{\partial t^2}) - \frac{T}{\rho} (\frac{\partial^2 u}{\partial x^2}) = 0, \tag{3}$$

which is the familiar wave equation for a continuous string. The action principle thus delivers the equations of motions for a continuous system, just as it does for a discrete set of masses.

Hamiltonian In order to do (non-path integral) quantum mechanics, we need the Hamiltonian of the system. To see how to construct the Hamiltonian for the continuous string, we start by going back to the discrete string. From Eq.(1) we define the momentum conjugate to $u(x)$,

$$p(x) \equiv \frac{\partial L}{\partial(\frac{\partial u(x)}{\partial t})} = \rho \Delta x \frac{\partial u(x)}{\partial t}.$$

Note that $p(x)$ is just the mass at x times the velocity. Dividing out the factor Δx in $p(x)$, we introduce the momentum *density*,

$$\pi(x) \equiv \rho \frac{\partial u(x)}{\partial t}.$$

Constructing the Hamiltonian for the discrete system in the standard way, we have

$$H = \sum_x p(x) \frac{\partial u(x,t)}{\partial t} - L = \sum_x \left(\frac{p(x)^2}{2\rho \Delta x} + \frac{T}{2} (\frac{\Delta u}{\Delta x})^2 \Delta x \right)$$

The first term, with Δx in the denominator, looks strange. However, if we express it using the momentum density, we obtain

$$H = \sum_x \Delta x \left(\frac{\pi^2}{2\rho} + \frac{T}{2} (\frac{\Delta u}{\Delta x})^2 \right)$$

Now it is clear how to take the continuum limit. We obtain for the Hamiltonian of the continuous string,

$$H = \int_0^{L_x} dx \left(\frac{1}{2\rho} \pi^2(x) + \frac{T}{2} (\frac{\partial u}{\partial x})^2 \right). \tag{4}$$

Just as for the Lagrangian, the Hamiltonian is the integral of a density for a continuous system.

Quantization What we have done so far is entirely classical. It merely illustrates how to construct a continuous or field system out of a discrete one. Now we want to proceed to quantize our system. As before, we start with the discrete system, where we know exactly what to do. For each coordinate $u(x,t)$, we impose standard quantum commutation rules with the corresponding conjugate momentum. This reads

$$[p(x,t), u(x',t)] = \frac{\hbar}{i} \delta_{x,x'}.$$

We have already seen that the natural quantity to make use of in the continuum limit is $\pi(x)$, not $p(x)$. Still remaining with the discrete system we write

$$[\pi(x,t), u(x',t)] = \frac{\hbar}{i} (\frac{\delta_{x,x'}}{\Delta x}).$$

Now for the continuum limit it is natural to replace

$$(\frac{\delta_{x,x'}}{\Delta x}) \text{ as } \Delta x \to 0$$

by the Dirac δ function,

$$\delta(x - x').$$

We finally have for the continuum form of the commutation rules,

$$[\pi(x,t), u(x',t] = \frac{\hbar}{i}\delta(x - x'), \tag{5}$$

so the field variable $u(x,t)$ and the momentum density $\pi(x,t)$ are the generalizations of coordinate and conjugate momentum for a discrete system, and their commutation rule Eq.(5) is the natural generalization of the familiar $[p,q]$ commutator of ordinary quantum mechanics.

To summarize, in the continuum system, the Lagrangian and Hamiltonian are given by Eq.(2) and Eq.(4), the equation of motion is Eq.(3), and the quantum commutation rule is Eq.(5).

Harmonic Oscillators and Normal Modes Our system, either discrete or continuous, has a Lagrangian which is quadratic. Such systems always reduce to a set of harmonic oscillators. For our continuous string, with *fixed* ends, we introduce an expansion in normal modes as follows,

$$u(x,t) = \sum_k u_k(t)\phi_k(x), \tag{6}$$

and

$$\pi(x,t) = \sum_k \pi_k(t)\phi_k(x),$$

where the $\phi_k(x)$ are

$$\phi_k(x) \equiv \sqrt{\frac{2}{L_x}}\sin(kx),$$

and the wave numbers k are given by

$$k = \frac{n\pi}{L_x}, \quad n = 1, 2, \ldots.$$

The ϕ_k are a complete, orthonormal set of functions. The orthogonality condition is

$$\int_0^L dx \phi_k(x)\phi_{k'}(x) = \delta_{k,k'}$$

The normal mode coordinates are the $u_k(t)$. To find the normal mode frequencies, we use the equation of motion,

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}.$$

Substituting the normal mode expansion, we have

$$\sum_k (-\omega_k^2) u_k \phi_k(x) = \sum_k \frac{T}{\rho}(-k^2) u_k \phi_k(x).$$

This equation demands that

$$\omega_k^2 = c^2 k^2, \quad \text{where} \quad c^2 = \frac{T}{\rho}$$

Our next move is to rewrite the Lagrangian in terms of the u_k. Taking first the kinetic energy, we substitute the mode expansion of u from Eq.(6) and use the orthogonality of the ϕ_k, and obtain

$$K = \int_0^L dx \frac{\rho}{2} \left(\sum_k \frac{\partial u_k}{\partial t} \phi_k(x) \right) \left(\sum_{k'} \frac{\partial u_{k'}}{\partial t} \phi_{k'}(x) \right) = \sum_k \frac{\rho}{2} (\frac{\partial u_k}{\partial t})^2$$

The treatment of the potential is similar. Again orthogonality is used and gives

$$V = \frac{2}{L} \int_0^L dx \frac{T}{2} \left(\sum_k u_k(t) k \cos(kx) \right) \left(\sum_{k'} u_{k'}(t) k' \cos(k'x) \right) = \sum_k \frac{T}{2} k^2 u_k^2.$$

We finally have for the Lagrangian,

$$L = K - V = \sum_k \frac{\rho}{2} \left((\frac{\partial u_k}{\partial t})^2 - \omega_k^2 u_k^2 \right),$$

which is clearly a sum of independent harmonic oscillators, each with its own frequency.

Creation and Destruction Operators Having seen that the Lagrangian is a sum of oscillators, we introduce creation and destruction operators for each mode, setting

$$u_k = \sqrt{\frac{\hbar}{2\rho\omega_k}} (a_k + a_k^\dagger),$$

and

$$\pi_k = -i\sqrt{\frac{\hbar\rho\omega_k}{2}} (a_k - a_k^\dagger).$$

The a_k and $a_k^\dagger$ obey the usual commutation rules,

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'} \tag{7}$$

Writing out the expansion of u and π, we now have

$$u(x,t) = \sum_k \sqrt{\frac{\hbar}{2\rho\omega_k}} \left(a_k(t) + a_k^\dagger(t) \right) \sqrt{\frac{2}{L_x}} \sin(kx),$$

and

$$\pi(x,t) = -i\sum_k \sqrt{\frac{\hbar\rho\omega_k}{2}} \left(a_k(t) - a_k^\dagger(t)\right) \sqrt{\frac{2}{L_x}} \sin(kx).$$

A consistency check is the $\pi(x)$ and $u(x)$ commutator. We have using Eq.(7)

$$[\pi(x,t), u(x',t)] = \frac{\hbar}{i}\sum_k \phi_k(x)\phi_k(x') = \frac{\hbar}{i}\delta(x-x'),$$

where the last equality holds because the ϕ_k are a complete set. For the Hamiltonian, returning to

$$H = \int_0^{L_x} dx \left(\frac{1}{2\rho}\pi^2(x) + \frac{T}{2}(\frac{\partial u}{\partial x})^2\right),$$

substituting for u and π and carrying out the integral over x, we get

$$H = \sum_k (\hbar\omega_k(a_k^\dagger a_k + \frac{1}{2}).$$

This is just the usual formula for a set of oscillators. Note that since there is no upper limit on k, the sum over zero point energies diverges. This quantity although divergent, is a constant, and is omitted by redefining the zero of energy. Given the form of the Hamiltonian, it is easy to figure out the Heisenberg form of the creation and destruction operators,

$$a_k(t) = \exp(\frac{iHt}{\hbar})a_k \exp(\frac{-iHt}{\hbar}) = \exp(-i\omega_k t)a_k,$$

and

$$a_k^\dagger(t) = \exp(\frac{iHt}{\hbar})a_k^\dagger \exp(\frac{-iHt}{\hbar}) = \exp(i\omega_k t)a_k^\dagger$$

Using these in the expansions of u and π, we finally have

$$u(x,t) = \sum_k \sqrt{\frac{\hbar}{2\rho\omega_k}} \left(a_k e^{-i\omega_k t} + a_k^\dagger e^{i\omega_k t}\right) \sqrt{\frac{2}{L_x}} \sin(kx) \tag{8}$$

and

$$\pi(x,t) = -i\sum_k \sqrt{\frac{\hbar\rho\omega_k}{2}} \left(a_k e^{-i\omega_k t} - a_k^\dagger e^{i\omega_k t}\right) \sqrt{\frac{2}{L_x}} \sin(kx) \tag{9}$$

Eqs.(7) and (9) are characteristic of expansions of fields; there is a "destruction" part and a "creation" part, each of which moves with its frequency. If we were to couple a system to the field u, we could then describe the emission and absorption of normal modes of our string, which would be a type of phonon.

Periodic Boundary Conditions Although field systems with fixed boundary conditions can be of interest, it is more typical that the system is put in a "box" with periodic boundary conditions, and the "volume" of the box is taken to ∞ at the end of the calculation. For our string, the form of the continuum Lagrangian, Hamiltonian, equation of motion, and commutation rules for periodic boundary conditions are still given by Eqs.(2),(4),(3), and (5), respectively. What changes is the mode expansion. The appropriate functions are plane waves,

$$\frac{1}{\sqrt{L_x}}\exp(ikx), \quad k=\frac{2\pi n}{L_x}, \quad n=0,\pm1,\pm2,\ldots$$

These plane waves form a complete, orthonormal set. The non-zero k values all have independent harmonic oscillators associated with them with corresponding destruction and creation operators a_k and $a_k^\dagger$. Since there is no point of the string which is held fixed, there is of course no oscillator associated with $k=0$. This is a so-called "zero mode" which for a string would have to be handled separately. It corresponds to the overall motion of the center of mass of the string. The physics is usually in the "relative" or oscillator part of the system. Omitting the zero mode, the field expansions are now

$$u(x,t)=\sum_k{}' \sqrt{\frac{\hbar}{2L_x\rho\omega_k}}\left(a_k e^{ikx-i\omega_k t}+a_k^\dagger e^{-ikx+i\omega_k t}\right) \tag{10}$$

and

$$\pi(x,t)=-i\sum_k{}' \sqrt{\frac{\hbar\rho\omega_k}{2L_x}}\left(a_k e^{ikx-i\omega_k t}-a_k^\dagger e^{-ikx+i\omega_k t}\right) \tag{11}$$

The oscillator part of the Hamiltonian is

$$H=\sum_k{}'(\hbar\omega_k(a_k^\dagger a_k+\frac{1}{2}).$$

The frequencies are still

$$\omega_k^2=c^2k^2, \quad \text{where } \; c^2=\frac{T}{\rho}$$

Overall, there the expansions for fixed and periodic boundary conditions are rather similar, the main difference being the substitution of plane waves for the $\phi_k(x)$, and the formulae for the allowed values of the wave number k.