Solution to HW2

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1 Symmetry of an action

Throughout this problem, we ignore the Poincare symmetry (translation + Lorentz).

1.1

This action has only a Z_2 symmetry: $a_2 \to -a_2$, which is not a continuous symmetry, so there's no conserved current.

1.2

Let $\phi = a_1 + ia_2$, the Lagrangian then becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi^* - m^2|\phi|^2 - \lambda|\phi|^4),$$

where $|\phi|^2 = \phi^*\phi$. We can see that changing the phase of the complex field doesn't change the action. So the action has a $U(1)$ symmetry(this symmetry can also be thought of as the rotation between a_1 and a_2, which is an SO(2) symmetry). The corresponding transformation is $\delta a_1 = \epsilon a_2$, $\delta a_2 = -\epsilon a_1$. The corresponding conserved current is thus

$$j_\mu = a_1 \partial_\mu a_2 - a_2 \partial_\mu a_1$$

1.3

If we write $\phi = \rho e^{i\theta}$, then the term $a_1^2 a_2^2 \propto \sin 2\theta$, which break the U(1) symmetry. So there's no continuous symmetry.

1.4

Let $\phi = a_1 + ia_2$, $\chi = a_3 + ia_4$, we then have

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi^* - m^2|\phi|^2 + \partial_\mu \chi \partial^\mu \chi^* - m^2|\phi|^2 - \nu^2 \Re[\phi\chi]),$$

where $\Re[z]$ means the real part of z. There's a U(1) symmetry when we transform $\phi \to e^{i\alpha}\phi$, $\chi \to e^{-i\alpha}\chi$, or, written in infinitesimal form,

$$\delta a_1 = \epsilon a_2,\ a_2 = -\epsilon a_1; \quad a_3 = -\epsilon a_4,\ a_4 = \epsilon a_3.$$

The corresponding conserved current is

$$j_\mu = a_1 \partial_\mu a_2 - a_2 \partial_\mu a_1 - a_3 \partial_\mu a_4 + a_4 \partial_\mu a_3.$$

2 Charges as generator of a symmetry

2.1

Let $A = iQ_k \epsilon^k$, $B = \psi^b(x)$, and insert into the formula given in the problem, we have

$$\begin{aligned}
&[A, B] = i\epsilon^k [Q_k, \psi^b(x)] = i\epsilon^k \lambda_a^b \psi^a(x), \\
&[A, [A, B]] = (i\epsilon^l) * (i\epsilon^k)[Q_l, \lambda_a^b \psi^a(x)] = (i\epsilon^l \lambda_l)_a^{2b} \psi^a \\
&......
\end{aligned}$$

We thus prove that $e^{i\epsilon^k Q_k} \psi_b(x) e^{-i\epsilon^k Q_k} = \left(e^{i\epsilon^k \lambda_k}\right)_b^a \psi_a$.

2.2

$$t_1 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; t_2 = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; t_3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2.3

For the infinitesimal transformation $\delta\psi_a = (t_k)_a^b \psi_b$, the conserved charge desity is

$$J_k^0(\mathbf{x}, t) = \pi_a(\mathbf{x}, t)\,(t_k)^a{}_b\,\psi_b(\mathbf{x}, t). \tag{1}$$

Compute the commutator

$$[J_k^0(\mathbf{x}), J_\ell^0(\mathbf{y})] = [\pi_a(\mathbf{x})(t_k)^a{}_b\,\psi_b(\mathbf{x}),\; \pi_c(\mathbf{y})(t_\ell)^c{}_d\,\psi_d(\mathbf{y})]\,. \tag{2}$$

Only terms where a momentum encounters a field at the same time and location contribute. We get

$$\begin{aligned}[J_k^0(\mathbf{x}), J_\ell^0(\mathbf{y})] &= \pi_a(\mathbf{x})(t_k)^a{}_b\,[\psi_b(\mathbf{x}), \pi_c(\mathbf{y})]\,(t_\ell)^c{}_d\,\psi_d(\mathbf{y}) \\ &\quad - \pi_c(\mathbf{y})(t_\ell)^c{}_d\,[\psi_d(\mathbf{y}), \pi_a(\mathbf{x})]\,(t_k)^a{}_b\,\psi_b(\mathbf{x}).\end{aligned} \tag{3}$$

Using $[\psi_b(\mathbf{x}), \pi_c(\mathbf{y})] = i\delta_{bc}\delta^3(\mathbf{x}-\mathbf{y})$ and $[\psi_d(\mathbf{y}), \pi_a(\mathbf{x})] = -i\delta_{da}\delta^3(\mathbf{x}-\mathbf{y})$, this becomes

$$\begin{aligned}[J_k^0(\mathbf{x}), J_\ell^0(\mathbf{y})] &= i\,\pi_a(\mathbf{x})(t_k)^a{}_b(t_\ell)^b{}_d\,\psi_d(\mathbf{y})\,\delta^3(\mathbf{x}-\mathbf{y}) \\ &\quad + i\,\pi_c(\mathbf{y})(t_\ell)^c{}_d(t_k)^d{}_b\,\psi_b(\mathbf{x})\,\delta^3(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{4}$$

Using the delta to set $\mathbf{y} \to \mathbf{x}$ inside operators and relabeling indices, we obtain

$$\begin{aligned}[J_k^0(\mathbf{x}), J_\ell^0(\mathbf{y})] &= i\,\pi_a(\mathbf{x})\big[(t_k t_\ell) - (t_\ell t_k)\big]^a{}_b\,\psi_b(\mathbf{x})\,\delta^3(\mathbf{x}-\mathbf{y}) \\ &= i\,\pi_a(\mathbf{x})[t_k, t_\ell]^a{}_b\,\psi_b(\mathbf{x})\,\delta^3(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{5}$$

Using the Lie algebra $[t_k, t_\ell] = i\,\varepsilon_{k\ell m} t_m$, we get

$$[J_k^0(\mathbf{x}), J_\ell^0(\mathbf{y})] = i\,\varepsilon_{k\ell m}\,J_m^0(\mathbf{x})\,\delta^3(\mathbf{x}-\mathbf{y}) \tag{6}$$

3 Casimir Energy

3.1 (a) and (b)

Fourier expand

$$\phi(x, t) = \sum_{n\in\mathbb{Z}} \phi_n(t)\,e^{ip_n x}, \tag{7}$$

with periodicity requiring

$$e^{ip_nL} = 1 \quad \Rightarrow \quad p_n = \frac{2\pi n}{L}, \qquad n \in \mathbb{Z}. \tag{8}$$

For a massless field, $\omega_n = |p_n| = 2\pi|n|/L$. A real field implies $\phi_{-n} = \phi_n^*$, and the vacuum energy (zero-point energy) is formally

$$\begin{aligned} E_0 &= \sum_{n\neq 0} \frac{1}{2}\,\omega_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\omega_n + \frac{1}{2}\omega_{-n}\right) \\ &= \sum_{n=1}^{\infty} \omega_n = \sum_{n=1}^{\infty} \frac{2\pi n}{L}. \end{aligned} \tag{9}$$

Often one rewrites this as

$$E_0 = \frac{\pi}{L}\sum_{n=1}^{\infty} n, \tag{10}$$

absorbing conventions into the overall factor. Thus in the notation $E_0 = \#\sum_{n=1}^{\infty} n$, we have

$$\# = \frac{\pi}{L}. \tag{11}$$

3.2 regulation

We can consider the regulated energy

$$E_0^{\Lambda} = \frac{\pi}{L}\sum_{n=1}^{\infty} n e^{n/\Lambda L} \tag{12}$$

Recall that

$$\sum_{n=1}^{\infty} n e^{-an} = -\frac{d}{da}\sum_{n=1}^{\infty} e^{-an} = -\frac{d}{da}\frac{e^{-a}}{1-e^{-a}} = \frac{e^{-a}}{(1-e^{-a})^2}.$$

We thus have

$$E_0^{\Lambda} = \frac{\pi}{L}\frac{e^{-1/\Lambda L}}{(1-e^{-1/\Lambda L})^2} \tag{13}$$

Let $x = \frac{1}{\Lambda L}$, when $\Lambda \to \infty$, $\alpha \to 0$,

$$\frac{e^{-x}}{(1-e^{-x})^2} = \frac{1}{x^2} - \frac{1}{12} + o(x^2) \tag{14}$$

The cutoff independent part is $-\frac{1}{12}$.

3.3 Casimir effect

$\rho = \frac{E}{L} = -\frac{\pi}{L^2}$, $\frac{\partial \rho}{\partial L} = \frac{\pi}{6L^3}$. This is the Casimir force on unit area. (Note that the leading term in (14) divided by L doesn't depend on L.

4 Propagator between two mirrors

Recall that when computing the electric potential (or thus the electric field) between the two charges in a box, we usually "remove" the box by adding an infinite number of charges such that at the location of the box, the potential is zero. We can do similar thing here. Suppose we have a source placed at y, to make the potential at x=0 zero, we have to put a negative source at $x = -y$, similarly, we have to put a negative source at $x = 2L - y$. Next, to cancel the potential induced by these two negative source, we have to put the negative source at $2L + y$ and $y - 2L$. If we repeat the process we have positive sources on $y + 2nL$, negative sources on $y - 2nL$, where $n \in \mathbb{Z}$. So we have

$$G_{\text{box}}(x, y) = \sum_{n\in\mathbb{Z}} G_{\text{Free}}(x, y + 2nL) - G_{\text{Free}}(x, -y + 2nL),$$

where $G_{\text{Free}}(x, y)$ is the propagator for a free massless scalar field between x and y.

5 Local QFT

Consider the Fourier decomposition of the quantum field,

$$\phi(\mathbf{x}, t) = \int \frac{d^3x}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \tilde{\phi}(\mathbf{k}, t).$$

In momentum space, the equation of motion then becomes

$$(\partial_t^2 + k^2 + m^2)\tilde{\phi}(\mathbf{k}) = 0.$$

Let $\omega_k = k^2 + m^2$, and use the initial condiiton, we then have

$$\tilde{\phi}(\mathbf{k}, t) = \tilde{\phi}(\mathbf{k}, 0) \cos \omega_k t + \tilde{\pi}(\mathbf{k}, 0) \frac{\sin \omega_k t}{\omega_k},$$

Where$\tilde{\phi}(\mathbf{k}, 0) \cos \omega_k t$, $\tilde{\pi}(\mathbf{k}, 0)$ are the Fourier transformation of the $\tilde{\phi}(\mathbf{k}, 0)$ and $\pi(\mathbf{k}, 0)$ respectively. We thus have

$$\phi(\mathbf{x}, t) = \int d^3k e^{i\mathbf{k}\cdot\mathbf{x}} \left[\tilde{\phi}(\mathbf{k}, 0) \cos(\omega_{\mathbf{k}} t) + \tilde{\pi}(\mathbf{k}, 0) \frac{\sin(\omega_{\mathbf{k}} t)}{\omega_{\mathbf{k}}}\right]. \tag{15}$$

Denote $\Delta(x,t;y,t') = \int \frac{d^3k}{(2\pi)^3} e^{-ik(x-y)} \frac{\sin \omega_k(t-t')}{\omega_k}$, we then have

$$\phi(\mathbf{x},t) = \int d^3y \Delta(x,t;y,0)\pi(\mathbf{y},0) + \dot{\Delta}(x,t;y,0)\phi(\mathbf{y},0) \tag{16}$$

Consider two fields $\phi(\mathbf{x},t)$, $\phi(\mathbf{y},t')$. If they're spacelikely separated, by performing a boost, we can transform them into a new coordinate where they are at the same time slice, the kernel then vanishes, so $[\phi(\mathbf{x},t), \pi(\mathbf{y},t')]$ will then vanish, which means they're not causally related.