Topological origin of equatorial waves


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Outline

❖ **Background**
❖ Summary
❖ Critical evaluation
❖ Citations and future work
❖ Conclusions
Classical methods are sufficient to characterize equatorial waves

- With great effort

- Only rotating-shallow-water equations required
  

- Atmospheric observations have since vindicated this work
  

Dispersion relations of various equatorial wave modes
Topological methods have been applied to hydrodynamics

- Mostly in the context of dynamo theory or magnetohydrodynamics
- Protected edge states not considered


The Earth’s mantle is an application of topology to hydrodynamics.
Atmospheres have been treated as condensed matter

- Author also co-wrote “Topological origin”
- Built condensed matter models of planetary atmospheres
- Does not discuss topology


Topological methods have great predictive power for atmospheric quantities like vorticity
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Coriolis force causes equatorial waves

- Ocean and atmospheric waves trapped close to the equator
- Rapid decay away from the equator due to Coriolis force
- Spherical shape of the earth increases the magnitude of the Coriolis force away from the equator

\[
\mathbf{a}_C = 2v \times \Omega \\
\mathbf{F}_C = 2m v \times \Omega
\]

_Coriolis acceleration_  
_Coriolis force_

Kelvin and Yanai (Rossby-Gravity) waves have been studied previously

- Propagate energy eastward along the equator
- Kelvin modes travel eastward
- Yanai modes can travel westward given periods are substantially long
- Contribute to earth’s climate dynamics
  - El Niño-Southern oscillation
  - Quasi-biennial oscillation in the stratosphere
  - Madden-Julian Oscillation in the troposphere
  - Monsoons

https://earthobservatory.nasa.gov/images/43105/kelvin-wave-renews-el-niao
El Niño-Southern Oscillation is a Kelvin Wave

- Warm water is transferred across the Pacific to South America
- Causes extreme weather events
- Excitations in the Indian ocean excite a Kelvin Wave
- Kelvin Wave travels across the Pacific in 4 months

Phytoplankton in January immediately after El Niño, and in July
Kelvin and Yanai waves can be derived using shallow-water equations

- β-plane approximation: takes Coriolis parameter to vary linearly in space (f = βy)

\[
\begin{align*}
\frac{\partial u}{\partial t} - fv + g \frac{\partial h}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + fu + g \frac{\partial h}{\partial y} &= 0 \\
\frac{\partial h}{\partial t} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0
\end{align*}
\]

- $H$ : depth of the fluid
- $x$ : zonal (horizontal) direction
- $y$ : meridional direction
- $u, v$ : $x$ and $y$ fluid velocities
Kelvin and Yanai waves can be derived using shallow-water equations

- Two of the solutions have eastward group velocity

\[ u = \hat{u}(y) e^{i(kx + \sigma t)} \]
\[ v = \hat{v}(y) e^{i(kx + \sigma t)} \]
\[ \phi = \hat{\phi}(y) e^{i(kx + \sigma t)} \]

Kelvin: \[ \sigma = -k \]

Yanai: \[ \sigma = \sqrt{\left(\frac{k}{2}\right)^2 + 1 - \frac{k}{2}} \]
Kelvin Waves are Observed in the Oceans

Evidence of Kelvin waves
Solving the Wave Equation

$$\partial_t h + \nabla \cdot (hu) = 0$$
$$\partial_t u + (u \cdot \nabla) u = -g \nabla h - f \hat{n} \times u$$

$$f = 2\Omega \cdot \hat{n}$$
Solving the Wave Equation

\[ \partial_t h + \nabla \cdot (h \mathbf{u}) = 0 \]

\[ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -g \nabla h - f \hat{n} \times \mathbf{u} \]

Total time derivative of velocity field (acceleration)

\[ f = 2\Omega \cdot \hat{n} \]
Solving the Wave Equation

\[ \partial_t h + \nabla \cdot (hu) = 0 \]
\[ \partial_t u + (u \cdot \nabla) u = \begin{bmatrix} -g \nabla h \end{bmatrix} - f \hat{n} \times u \]

Force of gravity

\[ f = 2\Omega \cdot \hat{n} \]
Solving the Wave Equation

\[ \partial_t h + \nabla \cdot (h u) = 0 \]
\[ \partial_t u + (u \cdot \nabla) u = -g \nabla h - f \hat{n} \times u \]

Coriolis force

\[ f = 2\Omega \cdot \hat{n} \]
Finding Bulk Solutions

- Linearizing the equations gives a Schrödinger Equation
- The planewave solutions satisfy a dispersion relation

\[ \omega = \pm \sqrt{f^2 + c^2 k^2} \]

\[ i \partial_t \Psi = H \Psi \]

\[ \Psi = \begin{pmatrix} u_x \\ u_y \\ h - h_0 \end{pmatrix} \]

\[ \Psi = \Psi_0 e^{i(\omega t - k_x x - k_y y)} \]
Finding Solutions

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Coriolis effect breaks time-reversal invariance, causing the gap in the spectrum
Topologically protected edge states are nothing new

- Predicted in 1987, physically realized in 2008
- Bulk-boundary correspondence well established and not controversial
- Still somewhat popular to this day

Charles Kane and Joel Moore (2011). *Phys. World* 24 (02) 32
Patching the bulk solutions together

- Planewave solutions are good on a patch of the sphere with constant coriolis parameter \( f = 2\Omega \cdot \hat{n} \)
- Full solution patches together solutions around the sphere
- This can’t be done consistently because the system has Chern number = 2
- Implies existence of two edge modes at the equator
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How does “Topological origin” compare to previous work?

- Applies ideas from topology in condensed matter to geophysics in a novel way
- Crucially exploits the bulk-boundary correspondence
- Successfully replicates vetted results from classical theory and observation
Our impressions

❖ Positives
  ➢ Novel application of techniques from condensed matter physics to other areas
  ➢ Advertisement for topological insulators

❖ Negatives
  ➢ There are exactly three equatorial wave modes
  ➢ Analysis of real wave functions could elucidate the mechanism that prefers eastward over westward group velocity
  ➢ Difficult to generalize to other fields
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Citation Evaluation

- The paper is cited by 10 documents
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Authors’ Conclusions

- The notion of topologically protected edge states extends naturally to oceanic waves
- Observed dispersion relations were replicated with ideas from topology
- These techniques can be generalized to even more hydrodynamical systems
Our conclusions

❖ Positives
  ➢ Innovative extension of condensed matter physics
  ➢ Serves as justification for further study of topological insulators

❖ Negatives
  ➢ Extensive work remains to make the result robust
  ➢ Generalizations to other systems may not be as straightforward as authors purport
Questions?