Ginzburg-Landau Theory: Simple Applications

References: de Gennes ch. 6, Tinkham ch. 4, AJL QL sect. 5.7

Landau-Lifshitz (1936): 2nd order phase transition describable in terms of order parameter η, which is zero above T_c but takes finite value below, tending smoothly to zero as $T \to T_c$ (Example: spontaneous magnetization $\mathbf{S}$ of Heisenberg ferromagnet). η may be real scalar, complex scalar, real vector (e.g., ferromagnet), etc., but crucial feature is that η transforms under some symmetry group G which commutes with $\hat{H}$ (at least in absence of external 'symmetry-breaking' fields, e.g., $\hat{H}$ of FM invariant under rotation of $\mathbf{S}$ in absence of external magnetic field). This has the consequence that form of free energy, as a function of η, must be so constructed as to be invariant under G. Consider specifically case of complex scalar (appropriate to GL theory), then F must be invariant under rotation in Argand plane $\Rightarrow$ in spatially uniform case, only even powers of $|\eta|^2$ can occur[1]: for small η, expand[2]

$$F(\eta : T) = F_0(T) + \alpha(T)|\eta|^2 + \frac{1}{2}\tilde{\beta}(T)|\eta|^4 + \mathcal{O}(|\eta|^6) \tag{1}$$

Above T_c equilibrium value of $\eta = 0 \Rightarrow \alpha, \beta > 0$. Below T_c, equilibrium value of η finite $\Rightarrow \alpha(T) < 0$. Simplest hypothesis:

$$\alpha(T) = \alpha_0(T - T_c),\ \tilde{\beta} \cong \tilde{\beta}(T_c) \equiv \beta_0 \tag{2}$$

In spatially varying case, expect one contribution to F is simply $\int F\big(\eta(\mathbf{r}) : T\big)\, d\mathbf{r}$, with $F(\eta : T)$ as above, but in addition, expect gradient terms. Again, from analyticity and invariance under $\eta \to \eta e^{i\phi} (\phi \neq f(\mathbf{r}))$ expect second-order (in η) gradient terms to be proportional to $|\boldsymbol{\nabla}\eta(\mathbf{r})|^2$: $F_{\text{grad}} = \gamma(T)|\boldsymbol{\nabla}\eta(\mathbf{r})|^2$. [Note we can't necessarily argue that gradient terms of order η^4 have this form: cf. below.] Thus, most general form of $F\big(\eta(\mathbf{r}) : T\big)$ for T near T_c is (since simplest hypothesis is $\gamma(T) \cong \gamma(T_c) \equiv \gamma_0$)

$$F\big(\eta(\mathbf{r}) : T\big) = F_0(T) + \int \left\{\alpha_0(T - T_c)|\eta(\mathbf{r})|^2 + \frac{1}{2}\beta_0|\eta(\mathbf{r})|^4 + \gamma_0|\boldsymbol{\nabla}\eta(\mathbf{r})|^2\right\} d\mathbf{r} \tag{3}$$

Note a characteristic length (healing length) is defined by $\xi(T) = [\gamma_0/|\alpha_0(T - T_c)|]^{1/2} \sim |T - T_c|^{-1/2}$. Also note that theory is invariant under a constant rescaling of η: $\{\eta(\mathbf{r}) \to q\eta(\mathbf{r}),\ q \neq f(\mathbf{r})\}$ provided coefficients $\alpha_0, \beta_0, \gamma_0$ rescaled appropriately, i.e. absolute value of η has no significance. GL just choose the normalization so as to make $\gamma_0 = \hbar^2/2m$.

GL: take $\eta(\mathbf{r})$ to be a Schrödinger-type wave function, denote $\psi(\mathbf{r})$. Then the analysis goes through as above, except that we expect that in a vector potential $\mathbf{A}$, $\boldsymbol{\nabla} \to (\boldsymbol{\nabla} + \frac{1}{i\hbar}e^*\mathbf{A}(\mathbf{r}))$, e^* = effective charge associated with $\psi(\mathbf{r})$. Anticipate result that $e^* = 2e$, and choose normalization of $\psi(\mathbf{r})$ so that $\gamma_0 = \hbar^2/2m$, m = single e^- mass. (Thus, $\int |\psi(\mathbf{r})|^2 d\mathbf{r} \neq 1$ in general!)

[1] Odd functions of $|\eta|$ are forbidden by the requirement of analyticity.

[2] The conventional notation for the coefficient of $|\eta|^4$ is $\beta(T)$: I add a tilde to avoid confusion with $\beta \equiv 1/k_BT$. (but drop it on β_0, since no such confusion is likely to arise there).

Thus, adding explicitly the magnetic field energy $\int \frac{1}{2}\mu_0 \mathbf{B}(\mathbf{r})^2 \, d\mathbf{r}$ ($\mathbf{B} \equiv$ curl $\mathbf{A}$) we get the 'canonical' GL free energy (Lecture 3.) (ψ now $\to \Psi$)

$$F[\{\Psi(\mathbf{r})\}, T] = F_0(T) + \tag{4}$$
$$\int \left\{\alpha(T)|\Psi(\mathbf{r})|^2 + \frac{1}{2}\beta|\Psi(\mathbf{r})|^4 + \frac{1}{2m}|(-i\hbar\nabla - 2e\mathbf{A}(\mathbf{r})\Psi(\mathbf{r})|^2 + \frac{1}{2}\mu_0^{-1}(\nabla \times \mathbf{A})^2\right\} d\mathbf{r}$$

with $\alpha(T) = \alpha_0(T - T_c)$, $\beta \cong \beta_0 =$ const. The expression for the electric current, within this theory, is obtained by requiring that setting $\delta F/\delta \mathbf{A}(\mathbf{r}) = 0$ should give Maxwell's equation, $\mathbf{j}(\mathbf{r}) = \nabla \times \mathbf{H}(\mathbf{r}) = \mu_0^{-1} \, (\nabla \times (\nabla \times \mathbf{A}(\mathbf{r})))$, thus

$$\mathbf{j}(\mathbf{r}) = \frac{e}{m}(\Psi^*(-i\hbar\nabla - 2e\mathbf{A})\Psi + \text{c.c.}) \tag{5}$$

Note in particular that if Ψ is constant in space, then $\mathbf{j}(\mathbf{r}) = -\frac{4e^2}{m}|\Psi|^2\mathbf{A}(\mathbf{r})$.

Derivation of GL theory from BCS theory

It turns out that such a derivation is possible, if (a) normal component is in equilibrium (with static lattice, walls, etc.), (b) 'number of Cooper pairs' is small, and (c) all variations in space are small on the scale of the Cooper pair radius.[3] In this case, the GL OP turns out to be, apart from the normalization which is a matter of convention, simply the COM Cooper pair wave function, i.e., the quantity

$$\langle \psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r}')\rangle_{\mathbf{r}=\mathbf{r}'=\mathbf{R}} \equiv F(\mathbf{R} : \boldsymbol{\rho})_{\boldsymbol{\rho}=0} \tag{6}$$

that is

$$\Psi_{\text{GL}}(\mathbf{r}) = \text{const} \cdot F(\mathbf{R}, \boldsymbol{\rho})_{\boldsymbol{\rho}=0,\, \mathbf{R}=\mathbf{r}} \tag{7}$$

Usual textbook derivation is via the GL equation which results from minimizing F with respect to $\Psi(\mathbf{r})$ (cf. Lecture 3). I prefer to derive F directly: go through argument explicitly for uniform case, but can be generalized provided scale of variation long compared to ξ_0.

We define provisionally (i.e., forgetting about normalization)

$$\Psi = \langle \psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\rangle \equiv \sum_{\mathbf{k}} F_{\mathbf{k}}, \qquad F_{\mathbf{k}} \equiv \langle a_{-\mathbf{k}\downarrow}a_{\mathbf{k}\uparrow}\rangle \tag{8}$$

Consider the total free energy at finite T as a function of Ψ, where as usual we subtract a term μN. We have

$$F = \langle \hat{T}_{\text{kin}}\rangle - \mu\langle \hat{N}\rangle - TS + \langle \hat{V}\rangle \equiv K + \langle V\rangle \tag{9}$$

In the BCS model ($V_{\mathbf{k}\mathbf{k}'} \equiv -|V_0|$ within shell, zero outside), we have simply

$$\langle V\rangle = -|V_0|\Big(\sum_{\mathbf{k}} F_{\mathbf{k}}\Big)^2 \equiv -|V_0|\,|\Psi|^2 \tag{10}$$

[3]condition usually automatically satisfied in limit $T \to T_c$ since pair radius remains $\mathcal{O}(\xi_0)$ while both $\xi(T)$ and $\lambda(T)$ diverge for $T \to T_c$

so it remains to find the dependence of $K \equiv \langle T_{\text{kin}} - \mu N\rangle - TS$ on Ψ, for arbitary values of Ψ. Easiest way is probably as follows:

We want to find the minimum value of K subject to a fixed value of $\Psi \equiv \sum_{\mathbf{k}} F_{\mathbf{k}}$. Introduce then a Lagrange multiplier λ and minimize the quantity

$$K - \lambda(\Psi + \Psi^*), \qquad (\equiv K - 2\lambda\Psi, \text{ if } \lambda \text{ real}) \tag{11}$$

It is clear that in the Anderson pseudospin picture this is equivalent to neglecting the 'spin-spin' interaction (i.e. V) but applying an external "transverse" field $\mathcal{H}_x \equiv \lambda$. The quantity λ then plays exactly the same role as Δ in the original calculation, e.g. the excitation energies $E_{\mathbf{k}}$ are given by $(\epsilon_k^2 + |\lambda|^2)^{1/2}$ and thus we can write

$$\Psi(\lambda) \equiv \sum_{\mathbf{k}} F_{\mathbf{k}} = \lambda \sum_{\mathbf{k}} (2E_{\mathbf{k}}(\lambda))^{-1} \tanh(\beta E_{\mathbf{k}}(\lambda)/2), \qquad E_k \equiv (\epsilon_{\mathbf{k}}^2 + |\lambda|^2)^{1/2} \tag{12}$$

Moreover, since we minimized $K - 2\lambda\Psi$ with respect to Ψ at fixed λ, we have $\partial K/\partial\Psi = 2\lambda$; and since $K(\lambda = 0)$ is just the normal-state value ($\Psi = 0$), we get for the Ψ-dependent part ΔK

$$\Delta K(\Psi) = 2\int_0^{\Psi} \lambda(\Psi')d\Psi' \tag{13}$$

in principle, equations (10), (12), and (13) allow us to obtain $F(\Psi : T)$ for any value of Ψ and T, not just in the GL regime. In general, however, the inversion of (12) becomes very messy. It is only straightforward near T_c where we may assume that not only Ψ but λ is small and expand $\Psi(\lambda)$ in λ: we find straightforwardly from (12)

$$\Psi(\lambda) = A(T)\lambda - B(T)|\lambda|^2\lambda \tag{14}$$

where[4]

$$A(T) \equiv \sum_{\mathbf{k}} (2\epsilon_{\mathbf{k}})^{-1} \tanh(\beta\epsilon_k/2) = \frac{1}{2}\frac{dn}{d\epsilon}\int_0^{\epsilon_c} \frac{\tanh(\beta\epsilon/2)}{\epsilon}d\epsilon = \frac{1}{2}\frac{dn}{d\epsilon}\ln(1.14\beta\epsilon_c) \tag{15}$$

$$B(T) \equiv \Big(\frac{d}{d|\lambda|^2}\sum_{\mathbf{k}} \big(2E_{\mathbf{k}}(\lambda)\big)^{-1}\tanh(\beta E_{\mathbf{k}}/2)\Big|_{\lambda\to 0} = \sum_{\mathbf{k}} \frac{d}{d(\epsilon_{\mathbf{k}}^2)}(2\epsilon_k)^{-1}\tanh(\beta\epsilon_k/2)$$

$$= \frac{-1}{2}\frac{dn}{d\epsilon}\int_0^{\epsilon_c}\frac{1}{\epsilon}\frac{d}{d\epsilon}\frac{\tanh\beta\epsilon/2}{2\epsilon}d\epsilon = -(\beta^2/8)\frac{1}{2}\frac{dn}{d\epsilon}\int_0^{\beta\epsilon_c} z^{-1}\frac{d}{dz}(\tanh z/z)dz \approx \dots \int_0^{\infty}$$

$$= \frac{1}{2}\frac{dn}{d\epsilon}\beta^2(7/8\pi)\zeta(3)$$

where $\zeta(3) \equiv \sum_{n=1}^{\infty} n^{-3} \cong 1\cdot 2$

Inverting relation (14) up to 3rd order in Ψ, we find

$$\lambda(\Psi) = A^{-1}\Psi + BA^{-4}|\Psi|^3 \tag{16}$$

[4] note $\sum_{\mathbf{k}}$(no $\sum_\sigma$!) and $\frac{1}{2}\frac{dn}{d\epsilon}, \int_{-\epsilon_c}^{\epsilon_c} d\epsilon$, $\frac{dE_k}{d(\lambda^2)} = \frac{dE_k}{d(\epsilon^2)} = \frac{1}{2\epsilon}\frac{dE}{d\epsilon}$

and inserting this into (13) gives

$$\Delta K(\Psi) = A^{-1}(T)|\Psi|^2 + \frac{1}{2}B(T)\,|A(T)|^{-4}|\Psi|^4 \tag{17}$$

Thus, adding the potential-energy term, we finally get for the free energy $F(\Psi : T)$

$$F(\Psi : T) = F_0(T) + \{A^{-1}(T) - |V_0|\}|\Psi|^2 + \frac{1}{2}B(T)|A(T)|^{-4}|\Psi|^4 \tag{18}$$

This is of the general form of the bulk terms in the GL free energy. It is clear that T_c is defined by the point at which the coefficient of $|\Psi|^2$ goes negative, i.e., at the temperature defined by

$$\frac{1}{2}dn/d\epsilon\ln(1.14\beta_c\epsilon_c) = 1/|V_0| \tag{19}$$

which is just the BCS equation for T_c. Eliminating $|V_0|$ and expanding for T close to T_c, we find

$$F(\Psi : T) = F_0(T) + (dA^{-1}/dT)_{T_c}(T - T_c)|\Psi|^2 + \frac{1}{2}B(T)|A(T)|^{-4}||\Psi|^4 \tag{20}$$

We could perfectly well use this expression as our GL free energy. However, it turns out to simplify the ensuing formulae somewhat if we introduce the quantity $|V_0|\Psi = [A(T_c)]^{-1}\Psi$ which since it has the dimensions of the energy gap Δ and reduces to it for homogeneous equilibrium (see below), we will denote $\tilde{\Delta}$: in terms of $\tilde{\Delta}$ we have (since $d(\ln\, A)/dT = -1/T \approx 1/T_c$)

$$F(\tilde{\Delta} : T) = F_0(T) + N(0)\big\{ - (1 - T/T_c)|\tilde{\Delta}|^2 + \frac{1}{2}\,\frac{7\zeta(3)}{8\pi^2}\,\frac{1}{(k_BT_c)^2}\,|\tilde{\Delta}|^4 + \dots\big\} \tag{21}$$

Needless to say, differentiation of $F(\tilde{\Delta} : T)$ with respect to $\tilde{\Delta}$ gives back the BCS result for $T \to T_c$, $\tilde{\Delta}(T) = 3.06k_{\mathrm{B}}T_c(1 - T/T_c)^{1/2}$ (so in this case $\tilde{\Delta} = \Delta$, the energy gap).

Now consider the gradient terms. The easiest way of deriving these is to consider a simple phase gradient and compare with the definition of the superfluid density. If we take $\mathbf{v}_s \equiv \frac{\hbar}{2m}\boldsymbol{\nabla}\phi$, there the phenomenological expression for the energy due to flow is:

$$\Delta F = \frac{1}{2}\rho_s(T)v_s^2 = \frac{\hbar^2}{8m^2}\rho_s(T)(\nabla\phi)^2 \tag{22}$$

while the GL expression is

$$\Delta F = \gamma(T)|\boldsymbol{\nabla}\psi|^2 = \gamma(T)|\Psi|^2(\boldsymbol{\nabla}\phi)^2 \tag{23}$$

Equating these two expressions gives

$$\gamma(T) = \frac{\hbar^2\rho_s(T)}{8m^2|\Psi|^2(T)} \tag{24}$$

which is of course valid for any normalization of Ψ. Suppose we choose the normalization so that $\Psi = \tilde{\Delta} \cong \Delta(T)$ (corrections are of higher order in the gradient term). Then we can use the result that for a pure system $\rho_s/\rho = 1 - Y(T) \cong \left(7\zeta(3)/(4\pi^2 k_B^2 T_c^2)\right)\Delta^2(T)$, so

$$\gamma(T) = \frac{n\hbar^2}{4m}\frac{7\zeta(3)}{8\pi^2 k_B^2 T_c^2} \tag{25}$$

(Note: independent of T for $T \to T_c$)

Inclusion of magnetic vector potential:

It is intuitively plausible that since Ψ is the wave function of the COM of a Cooper pair, in a magnetic vector potential the $\boldsymbol{\nabla}$ should be replaced by $\boldsymbol{\nabla} - 2ie\mathbf{A}(\mathbf{r})/\hbar$. Formally this follows because

$$\sum_i \frac{\mathbf{p}_i^2}{2m} \to \sum_i \left(\mathbf{p}_i - e\mathbf{A}(\mathbf{r}_i)\right)^2/2m \to \frac{1}{2m}\sum_i |(-i\hbar)\boldsymbol{\nabla}_i - e\mathbf{A}(\mathbf{r}_i)|^2 \tag{26}$$

Amplitude variations:

In the general case (Ψ not small) there is no good reason to suppose that the "bending energy" associated with variation in the amplitude should be simply related to that for phase variation (which we have just related to ρ_s). But for small Ψ, the analyticity argument indicates that the form $\gamma|\boldsymbol{\nabla}\Psi|^2$ is unique, justifying the GL term. Thus we finally can write our GL free energy in the form:

$$F(\Psi(\mathbf{r}), T) = F_0(T) + \tag{27}$$
$$\int \left\{\alpha(T)|\Psi(\mathbf{r})|^2 + \frac{1}{2}\tilde{\beta}(T)\left|\Psi(\mathbf{r})\right|^4 + \gamma(T)\left|(\boldsymbol{\nabla} - 2ie\mathbf{A}/\hbar)\Psi(\mathbf{r})\right|^2 + \frac{1}{2}\mu_0^{-1}(\boldsymbol{\nabla}\times\mathbf{A})^2\right\} d\mathbf{r}$$

and if we choose a normalization so that $\Psi(\mathbf{r})$ is equal to $\tilde{\Delta}(\mathbf{r})$, then the coefficients are given for the pure case as

$$\alpha(T) = -\frac{1}{2}(dn/d\epsilon)(1 - T/T_c) \tag{28}$$
$$\tilde{\beta}(T) = \frac{1}{2}(dn/d\epsilon)\frac{7\zeta(3)}{8\pi^2}\frac{1}{k_B^2 T_c^2} \sim \text{const.} \equiv \beta_0$$
$$\gamma(T) = \frac{n\hbar^2}{4m}(dn/d\epsilon)\frac{7\zeta(3)}{8\pi^2}\frac{1}{k_B^2 T_c^2} \sim \text{const.} \equiv \gamma_0$$

For the dirty case, the values of α and β are practically unchanged, but $\gamma(T)$, like $\rho_s(T)$ is multiplied by a factor $\sim (l/\xi_0) \ll 1$.

The GL equations: differentiating the GL free energy functionally with respect to Ψ, we obtain

$$\alpha(T)\Psi(\mathbf{r}) + \beta_0|\Psi(\mathbf{r})|^2\Psi(\mathbf{r}) - \gamma_0(\boldsymbol{\nabla} - 2ie\mathbf{A}/\hbar)^2\Psi = 0 \tag{29}$$

As already noticed, the functional differentation with respect to $\mathbf{A}(\mathbf{r})$ gives Maxwell's equation [in the Landau gauge, div $\mathbf{A} = 0$]

$$\nabla^2 \mathbf{A}(\mathbf{r}) = \mu_0 \mathbf{j}(\mathbf{r}) \tag{30}$$

provided that electric current $\mathbf{j}(\mathbf{r})$ is identified as

$$\mathbf{j}(\mathbf{r}) = \frac{e}{m}\big(\Psi^*(-i\hbar\boldsymbol{\nabla} - 2e\mathbf{A})\Psi + \text{c.c.}\big) \tag{31}$$

The GL differential equation, which is 2nd order, must be supplemented by appropriate boundary conditions. If the system is closed, then the current through the boundary must vanish, implying:

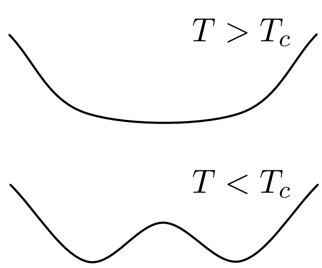
$$\mathbf{n}\cdot(i\hbar\boldsymbol{\nabla} - 2e\mathbf{A})\Psi = 0 \tag{32}$$

as boundary (Ψ itself need not vanish[5], cf. below).

Some simple solutions: experimental identification of GL parameters.

1. No magnetic field ($\mathbf{A} = 0, \psi(\mathbf{r}) =$ const in space)

Evidently, we then have $F = F_0 + V(\alpha(T)|\Psi|^2 + \frac{1}{2}\tilde{\beta}|\Psi|^4)\cdot[\alpha(T) \equiv \alpha_0(T - T_c)]$ (typical "Mexican-hat" potential). For $T > T_c, (\alpha(T) > 0)$, equilibrium value of $\Psi = 0, F = F_0(T)$. For $T < T_c$,



$$\Psi = -\alpha(T)/\beta_0 \tag{33}$$

$$F = F_n(T) - (1/2)\alpha^2(T)/\beta_0 = F_n(T) - \alpha_0(T_c - T)^2/2\beta_0$$

When we differentiate the free energy to get the entropy and the specific heat, the former is continuous at T_c but specific heat has a discontinuity due to the α-dependent term:

$$\Delta_v \equiv (c_s - c_n)\big|_{T=T_c} = -T\frac{d^2}{dT^2}\Big\{-\frac{\alpha_0(T_c - T)^2}{2\beta_0}\Big\} \tag{34}$$
$$\cong T_c\,\alpha_0^2/\beta_0$$

Thus the ratio α_0^2/β_0 (which is independent of the normalization of Ψ) is fixed by the experimental specific heat jump[6]. For a given material it is nearly independent of alloying (i.e. a weak concentration of impurities), as predicted by the microscopic theory.

2. Ψ varying in space but A negligible

One application is where no currents are flowing but $\Psi(\mathbf{r})$ is severely suppressed at some boundary, e.g. with ferromagnetic material. The GL equation reduces to (e.g., slab $\perp$

[5]On the scale of $\xi(T)$, etc. It must vanish on a scale $\sim k_F^{-1}$, because the single-particle wave functions making up the Cooper pair themselves vanish on this scale.

[6]In the literature the result is usually expressed in terms of the thermodynamic critical field $H_c(T)$

z-direction)

$$\alpha(T)\Psi(z) + \beta_0|\Psi(z)|^2\Psi(z) - (\gamma_0\partial^2/\partial z^2)\Psi(z) = 0, \quad \Psi(0) = 0$$
$$(T < T_c, \text{ i.e. } \alpha < 0) \tag{35}$$

The solution should clearly approach the equilibrium value $\Psi = \Psi_\infty \equiv -\alpha(T)/\beta_0$ at $z \to \infty$. It turns out to be

$$\Psi(z) = \Psi_\infty \tanh(z/\sqrt{2}\xi(T)) \tag{36}$$

where $\xi(T)$ is the GL correlation (coherence, healing) length previously introduced:

$$\xi(T) \equiv (\gamma/\alpha(T))^{1/2} \sim (T_c - T)^{-1/2} \tag{37}$$

(Note condition $\mathbf{n}\cdot\mathbf{j} = \mathbf{0}$, at walls automatically satisfied since Ψ is real!)

It should be strongly emphasized that contrary to what one might perhaps think there is no general veto on superconductivity occurring in a sample of dimension $\ll \xi(T)$ (provided boundary is e.g. with vacuum or insulator in which case Ψ does not need to vanish on a scale$\gg k_F^{-1}$, c.f. above), or even of dimension $\ll \xi_0$ (Cooper pair radius). The actual condition for complete suppression is more like $\Delta_0 <$ $(3D)$ single-particle energy splitting $\sim \epsilon_F/N^{2/3}$, which is a great deal more severe.

A second application: current-carrying state in a thin wire ($d \ll \lambda(T)$). Under this condition, $\mathbf{A}$ can be neglected to a first approximation and we have by symmetry $|\Psi| =$ const, $\Psi = |\Psi|e^{i\phi}$

$$\mathbf{j} = \frac{2e\hbar}{m}|\Psi|^2(\boldsymbol{\nabla}\phi) \tag{38}$$

or if we define $\mathbf{v}_s \equiv \frac{\hbar}{2m}\boldsymbol{\nabla}\phi$, $\mathbf{j} = 4e|\Psi|^2\mathbf{v}_s$. The energy is

$$F = -\alpha(T)|\Psi|^2 + \frac{\beta_0}{2}|\Psi|^4 + \gamma_0|\Psi|^2(\boldsymbol{\nabla}\phi)^2 \tag{39}$$

and minimizing this with respect to $|\Psi|$, we find

$$|\Psi| = \left(\frac{\alpha(T) - \gamma_0|\boldsymbol{\nabla}\phi|^2}{\beta_0}\right)^{1/2} \equiv |\Psi_\infty|\big(1 - \gamma_0|\boldsymbol{\nabla}\phi|^2/\alpha(T)\big)^{1/2}$$
$$\equiv (1 - \xi^2(T)(\boldsymbol{\nabla}\phi)^2)^{1/2}|\Psi_\infty| \tag{40}$$

Thus the OP vanishes when the "bending" of phase over a length $\xi(T)$ is equal to 1: at this point, the bending energy becomes equal to the original ($\Psi =$const) condensation energy. (It is therefore not surprising that if we extrapolate $\xi(T)$ to zero temperature, it is always of the order of the pair radius: $\xi(0) \sim \xi_0$ for a clean sample, $\sim (\xi_0 l)^{1/2}$ for a dirty one). Because of the relation (40), j is actually a nonmonotonic function of $\boldsymbol{\nabla}\phi(\mathbf{v_s})$, with a maximum at the point where $\boldsymbol{\nabla}\phi = \xi^{-1}(T)/\sqrt{3}$, i.e. $v_s = \frac{1}{\sqrt{3}}\frac{\hbar}{2m\xi}(|\Psi|^2 = (2/3)|\Psi_\infty|^2)$. Thus, the critical current is given by

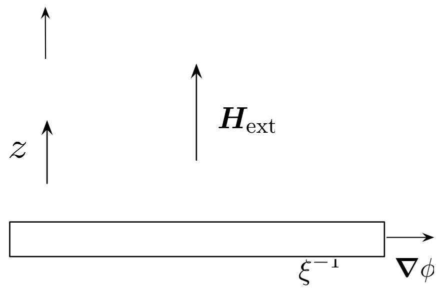
$$j_c = \frac{2e\hbar}{m}\frac{2}{3}\frac{\alpha(T)}{\beta_0}\frac{1}{\sqrt{3}}\xi^{-1}(T) \tag{41}$$

Since $\alpha(T) \sim T_c - T$ and $\xi(T) \sim (T_c - T)^{-1/2}$, this gives

$$j_c(T) \propto (1 - T/T_c)^{3/2} \tag{42}$$

Note: at low temperature, in particular as $T \to 0$, the behavior of $\mathbf{j}$ as a function of $\mathbf{v_s}$ is rather different. See Tinkham (1996) p. 125.

3. Isolated vortex line



Consider a thick (all dimension $\gg \lambda$) flat slab with normal in the z-direction, and the external field at $\infty \parallel z$.

One possibility: magnetic field is either totally excluded, or turns macroscropic regions of material normal + penetrates through these. This is indeed what happens in a type-I superconductor. (Tinkham, Section 5.1.)

However, there is a second possibility: the field may penetrate through in microscropic regions. Let's consider this possibility: field penetrates through a region of xy-plane centered at $\mathbf{r} = 0$ (convention!) while excluded from the bulk of the plane.

Consider a single such region. The field will by London's equation ($\mathbf{B} \sim \text{curl}\,\mathbf{j}$) produce a screening current which will flow around the edges of the "hole"; since situation qualitatively similar to bulk 2D, expect length over which current falls off is $\sim \lambda$. Beyond this point there is no current and no field, but there can still be a vector potential $\mathbf{A}$. Recall

$$\mathbf{j} \propto \boldsymbol{\nabla}\phi - \frac{2e}{\hbar}\mathbf{A} \tag{43}$$

so $j = 0 \Rightarrow \boldsymbol{\nabla}\phi = 2e\mathbf{A}$. But ϕ is gradient of phase of Ψ and thus must be single-valued modulo 2π, so

$$\oint \boldsymbol{\nabla}\phi \cdot d\boldsymbol{l} = 2n\pi \to \oint \mathbf{A} \cdot d\boldsymbol{l} \equiv \phi = n(h/2e) \equiv n\phi_0 \tag{44}$$

Hence, each such configuration ("vortex") encloses n flux quanta ϕ_0. Restrict consideration to $|n| = 1$, since $n = 0$ is trivial and $|n| \geq 2$ turns out to be unstable against decay into 2 or more $n = 1$ vortices. Since the extent of the (2D) field is $\sim \lambda$ the central field is $\sim \phi_0/\lambda^2$. Energy[7]: The energy is composed of an "intrinsic" energy E_0 necessary to form the vortex in zero field, plus the (negative) energy saved by "admitting" the external field, which is of order $\mu_0 H_{\text{ext}}\varphi_0/\lambda^2$. Consider E_0: this consists of (a) field energy, $\int(1/2)\mu_0 H^2 dr$, (b) (minus) condensation energy due to deviation of OP from

[7]Throughout this discussion "energy" means "energy per unit length."

bulk value, (c) flow energy $1/2\int \rho_s \mathbf{v}_s^2 d\mathbf{r}$ where $\mathbf{v_s} \equiv \frac{\hbar}{2m}(\boldsymbol{\nabla}\phi - 2(e/\hbar)\mathbf{A})$, (d) energy due to "bending" of amplitude of OP.

The order of magnitude of (a) is $(1/2)\mu_0 H_0^2 \lambda^2 \sim (\pi/2)\mu_0^{-1}\phi_0^2/\lambda^2$, (b) is negligible in the limit $\xi \ll \lambda$ (see below). (c) can be estimated from the consideration that $\mathbf{v}_s$ is of order $(\hbar/2m)(1/r)\partial\phi/\partial\theta \sim \hbar/2mr$ over a distance $r \sim \lambda$, thereafter tends to zero (Meissner). Thus

$$(\mathrm{c}) \sim \int (1/2)\rho_s \mathbf{v}_s^2 \, d^2r \sim (1/2)\rho_s(\hbar/2m)^2 2\pi \int_{r_0}^{\lambda} dr/r \sim (1/2)\rho_s(\hbar/2m)^2 \ln(\lambda/r_0). \quad (45)$$

($r_0 \sim$ lower cutoff). (d) is of order (c) without the logarithmic factor (in this case the upper limit on the r-integration is $\sim r_0$, c.f. below).

But we have the general relation (in the local limit).

$$\lambda^2 = (\mu_0 \rho_s e^2/m^2)^{-1} \quad (46)$$

so (c) can be written

$$(\mathrm{c}) \sim (1/2)(1/2\pi)(h/2e)^2(1/\mu_0\lambda^2)\ln(\lambda/r_0) \sim \mu_0^{-1}\frac{\phi_0}{\lambda^2}\ln(\lambda/r_0) \quad (47)$$

This is a larger than (a) by a logarithmic factor. Thus for $\lambda \gg r_0(\sim \xi$, see below) (c) is the dominant term and we find the "intrinsic" energy to create a vortex line to be

$$E_0 \sim \mu_0^{-1}(\phi_0^2/\lambda^2)\ln(\lambda/\xi), \quad (\lambda \gg \xi) \quad (48)$$

We have seen that the phase of the OP must satisfy $\oint \boldsymbol{\nabla}\phi \cdot d\boldsymbol{l} = 2\pi$ which means that for reasons of symmetry ϕ simply = polar "angle" θ. What about the magnitude of Ψ? To avoid a singularity we must have $\Psi = 0$ at $\mathbf{r} = 0$, i.e. at the vortex core. In a linear situation Ψ would heal to its equilibrium value Ψ_∞ over a distance $\sim \xi(T)$. For $\lambda \gg \xi$ this is still true for the vortex (because for $\xi \ll r \ll \lambda$ the bending term is only a fraction $\sim (\xi/r)^2$ of the condensation energy). Thus, Ψ relaxes to Ψ_∞ over a distance $\sim \xi(T)$, while $\mathbf{j}$ (or $\mathbf{v_s}$) relaxes to 0 over a distance $\sim \lambda(T)$. Since the energy (b) is of order $\pi\xi^2 E_{\mathrm{cond}} \sim \pi\xi^2\mu_0 H_c^2$, and $\mu_0 H_c^2 \sim \phi_0^2/\lambda^2\xi^2$ (cf. above), the term (b) is comparable to (a) and $\ll$ (c) in the extreme type-II limit $\lambda \gg \xi$.

A quantitative calculation based on the GL equations confirms those conclusions (see Tinkham Sections 5.1, or de Gennes Section 3.2). In addition, it may be shown that

(1) type-II behavior is only possible for $\kappa \equiv \lambda/\xi > 2^{-1/2}$.

(2) in the limit $\lambda \gg \xi$, the "lower critical field" at which penetration starts is $\sim \phi_0/\lambda^2$, more precisely

$$H_{c1} = (\phi_0/4\pi\lambda^2(T))\ln(\lambda/\xi) \equiv (\phi_0/4\pi\lambda^2(T))\ln\kappa \quad (+\text{ const.}) \quad (49)$$

(3) in the same limit, the "upper critical field" at which superconductivity is completely destroyed is $\sim \phi_0/\xi^2$, more precisely

$$H_{c2} = \phi_0/2\pi\xi^2(T) \quad (50)$$

For fields slightly less than H_{c2}, there are many ($\sim (\lambda/\xi)^2$) vortices within an area λ^2, i.e., the field is almost equal to its external value. It may be checked that if we neglect the ln in (49), we have the order of magnitude relation

$$H_{c1} H_{c2} \sim H_c^2 \tag{51}$$

where H_c is the thermodynamic critical field, as stated in lecture 2.