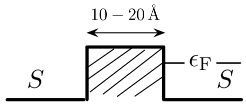
The Josephson Effect.

References: Tinkham (rev. ed.) ch. 6, sections 1–4; Barone & Paterno, Physics and Applications of the Josephson Effect, Wiley, New York, 1982

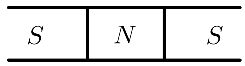
I will concentrate here on basics, and particularly on those aspects which will be most relevant to HTS. Josephson effect occurs whenever two bulk superconductors are connected via 'weak link', i.e., region which allows passage of electrons but with increased difficulty.

Examples:

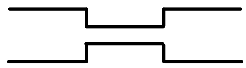
(1) tunnel oxide (S-I-S) junction: schematically,



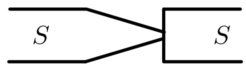
(2) proximity (S-N-S) junction



(3) constriction ('microbridge')



(4) point contact



Josephson's original prediction for such links, originally made specifically for case (1) but often (not always!) valid for others,[*] is 2 equations: If $\Delta\phi$ denotes the phase difference of the GL OP (i.e., of the Cooper pairs) across the junction, then

$$I = I_c \sin \Delta\phi \quad \text{dissipationless current} \tag{1}$$

$$\frac{d}{dt}\Delta\phi = \frac{2eV}{\hbar} \quad \text{where } V \text{ is voltage across junction} \tag{2}$$

Thus, for a finite dc voltage V applied across the junction, the current oscillates at a frequency $2eV/\hbar$:

$$I(t) = I_c \sin(2eVt/\hbar) \quad \text{(ac Josephson effect)} \tag{3}$$

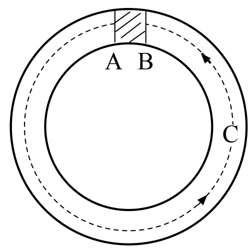
Actually, it is rather difficult to generate a constant dc voltage across the junction, and a more common situation is that of *constant current*: then $I(t) = \text{const} = I$, and provided $I < I_c$ we have from (1) $\Delta\phi = \text{const} = \sin^{-1} I/I_c$, whence from (2) $V = 0$ (compare case of bulk superconductor under constant-current conditions) (dc Josephson effect).

Significance of the Josephson effect: Depending on geometry, etc., I_c can range from $\sim$1 nA to 1 mA. We shall see that eqn. (1) implies a coupling energy of order $I_c\Phi_0/2\pi$; for most of the range this is small compared to $k_B T$ at room temperature. Yet this tiny energy can determine aspects of the behavior of circuits containing the junction, such as the trapped flux, which by any reasonable criterion are *macroscopic*! (Cf. recent SUNY and Delft experiments.)

[*]Eqn. (2) is very generally valid, but eqn. (1) less so.

It is possible to discuss the Josephson effect and related phenomena in terms of flux(oid) quantization,* provided one is willing to accept that the behavior of a junction in its own right should be independent of whether or not it forms part of a large superconducting ring. Consider a ring as shown, and assume that self-inductance effects can be neglected in this context: crudely speaking this will be so if $LI_c \ll \Phi_0$ where L is the self-inductance and I_c the critical current we are going to calculate. Under these conditions it does not actually matter whether the ring is thick or thin compared to the penetration depth λ, but it is convenient to assume the former. Consider now an external flux Φ applied through the ring under 'Aharonov-Bohm' conditions, i.e. such that the corresponding magnetic field is zero in the body of the ring. Because of the condition $LI_c \ll \Phi_0$, the flux through any circuit circumnavigating the ring, such as the contour shown, is simply Φ. Moreover, consider the expression for the electric current density:

$$\mathbf{j}(\mathbf{r}) \propto \boldsymbol{\nabla}\phi(\mathbf{r}) - 2e\mathbf{A}(\mathbf{r})/\hbar \tag{4}$$



where ϕ is the phase of the Cooper pairs and $\mathbf{A}(\mathbf{r})$ the vector potential. If we take C to be well inside the penetration depth, then $\mathbf{j}(\mathbf{r}) = 0$ and thus $\boldsymbol{\nabla}\phi(\mathbf{r}) = 2eA(\mathbf{r})/\hbar$. We can integrate this relation around the loop from A to B; since the width of the junction is negligible to the circumference of the ring, we can extend the integral over the RHS to run completely around the ring, whereupon it gives $\Phi(2e/\hbar)$.Thus, if $\Delta\phi$ denotes the difference between the phases of the Cooper pairs at A and B, we find

$$\boxed{\Delta\phi = 2\pi\Phi/\Phi_0} \quad \Phi_0 \equiv h/2e \tag{5}$$

This is a fundamental relation for superconducting circuits containing Josephson junctions. If we now differentiate it with respect to time and use Faraday's law, we get

$$\frac{d}{dt}\Delta\phi = \frac{2e}{\hbar}V(t) \tag{6}$$

where $V(t)$ is the emf developed around the ring, which can be interpreted as the voltage drop across the junction. Thus we immediately obtain the second Josephson equation (2).

Now let us consider the dependence of the free energy $F(\Phi, T)$ on the applied flux Φ. According to a standard result (cf. Problem 1.3) F must be periodic in Φ with period h/e, i.e., $2\Phi_0$. From time-reversal invariance, F must be an even function of Φ. Thus, we can make a Fourier expansion involving only cosines:

$$F(\Phi, T) = \sum_n A_n(T)\cos(2\pi n\Phi/2\Phi_0) = \sum_n A_n(T)\cos(n\Delta\phi/2) \tag{7}$$

This result is quite general and independent of the nature of the junction.

It is clear that in the limit of zero barrier transparency (e.g., a tunnel-oxide barrier with the (repulsive) potential V in the barrier region taken to ∞) $F(\Phi)$ is independent

*F. Bloch, Phys. Rev. B **2**, 109 (1970) of Φ, since under this condition the single-valuedness boundary condition is irrelevant (cf. Problem 1.3). Suppose now that the barrier (or more generally weak link) is indeed 'weak', in the sense that the matrix element for a single particle to traverse the barrier region, which we denote schematically[†] as t, may be treated as small and we may make a Taylor expansion of F in powers of t. Now, if we assume that the weak link is a small perturbation on the states of the bulk ring, then the effect of the external flux Φ is simply to multiply the single-particle wave functions at (say) B, and hence the value of the tunneling matrix t, by the factor $\exp ie\int \mathbf{A}d\boldsymbol{l} \equiv \exp i\Delta\phi/2$: that is,

$$t \to t' = t\exp[i(\Delta\phi/2)] \tag{8}$$

Consequently, the coefficient of the n-th term in the expansion (7) is of order t^n:

$$A_n(T) \sim t^n \tag{9}$$

We now come to a rather delicate point. Consider the odd terms in the expansion, e.g., for definiteness the $n = 1$ one, which, according to the above argument, is simply proportional to the single-particle tunneling matrix element t. Do we expect this to be nonzero? If we were talking about a single particle in the ring geometry, it most certainly would be, at least at $T = 0$. But for many particles obeying Fermi statistics, there is a strong argument that on average the effects are likely to be random between the different single-particle states and therefore to vanish in the limit of a large ring. This question has been of major interest in the last couple of decades in the 'mesoscopic' area; although not all aspects of theory and experiment are in agreement, both seem consistent with the hypothesis that single-particle (and more generally, 'odd-n') effects can be seen when the circumference $2\pi R$ of the ring is $\lesssim$ the so-called 'phase-breaking mean free path' l_ϕ (and in fact, evidence for a periodicity of $2\Phi_0$ is seen in some such (bulk) rings) but will vanish exponentially for $2\pi R \ll l_\phi$. Since, typically, $l_\phi \lesssim$ few thousand Å even at the lowest available temperatures, this means that those 'mesoscopic' single-electron effects should be negligible except in the very smallest (and usually purpose-built) rings. By contrast, the *even-n* effects are associated with the Cooper-pair wave function, which characterizes a single (COM) state and thus does not suffer cancellation.

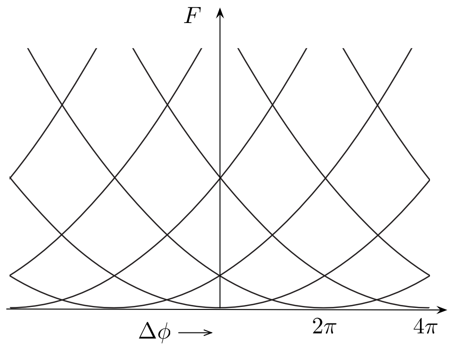
In the light of the above considerations, we can rewrite eqn. (7) with the notational change $2n \to n$:

$$\begin{aligned} F(\Delta\phi, T) &= \sum_{n=0}^{\infty} A_n(T)\cos(n\Delta\phi), \\ A_n(T) &\sim t^{2n} \end{aligned} \tag{10}$$

so that F is now indeed periodic in the Cooper-pair phase $\Delta\phi$ (rather than in $\Delta\phi/2$ as in the most general case). It should be emphasized that until we put some restrictions on the coefficients $A_n(T)$ the above expression can describe various qualitatively different types of behavior. In particular, it is entirely compatible with a *multivalued* form of $F(\Delta\phi)$, of which an extreme case is shown here: although any one 'branch' is not periodic in

[†]It will in general depend on the initial and final states, see below.

$\Delta\phi$, the pattern as a whole is. A useful distinction is based on the following thought-experiment: Suppose we start with $\Delta\phi = 0$ and gradually 'crank up' $\Delta\phi$, tracing out adiabatically a single branch of the function $F(\Delta\phi)$. When we get to π, one of two things can happen: either F can begin to fall again and eventually recover its $\Delta\phi = 0$ value at $\Delta\phi = 2\pi$, or it can go on rising. The first possibility corresponds to nonhysteretic behavior, the second to hysteresis (which may or may not persist up to $\Delta\phi = 2\pi$ and even far beyond). If one looks at the behavior of the pair wave function in coordinate space under the barrier, (in so far as it can be defined, cf. below) the second possibility corresponds to a form of $\Psi(z)$ at $\Delta\phi = \pi$ which is everywhere finite, (so that the cases $\Delta\phi = \pi + \varepsilon$ and $\Delta\phi = -\pi + \varepsilon$ are physically distinguishable), while the first corresponds to a node somewhere in the barrier region, so that one 'cannot tell' whether we have approached as it were from $\Delta\phi = 0$ or from $\Delta\phi = 2\pi$. The majority convention in the literature is probably to apply the name 'Josephson effect' only to the non hysteretic case, but this is not universal.



Clearly, if all the A_n's in eqn. (7) for $n \geq 2$ are negligible, as happens in the limit $t \to 0$ (see above) then we have the non hysteretic case, and this limit is often regarded as the 'pure' Josephson effect; from now on I will concentrate on this case unless otherwise noted. In this case we have a simple relation between the constant A_1 and the critical current I_c appearing in the first Josephson relation: Equating the change in the free energy of the junction to the work done by the voltage across the junction and using eqn. (5) gives

$$\frac{\partial F}{\partial(\Delta\phi)}\frac{d\Delta\phi}{dt} = IV = I\,\frac{\Phi_0}{2\pi}\,\frac{d\Delta\phi}{dt} \tag{11}$$

so

$$\partial F/\partial(\Delta\phi) = I\Phi_0/2\pi \tag{12}$$

or substituting for F from (10) and for I from the first Josephson relation,

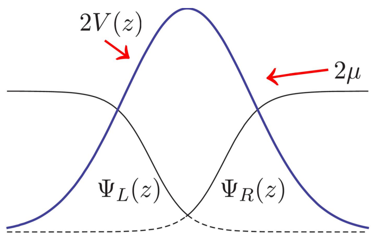
$$-A_1 \sin\Delta\phi = (I_c\Phi_0/2\pi)\sin\Delta\phi \tag{13}$$

so that $A_1 = -I_c\Phi_0/2\pi$, or in other words the free energy F of the junction is given in this case in terms of I_c by

$$\boxed{F(\Delta\phi) = (-I_c\Phi_0/2\pi)\cos\Delta\phi} \tag{14}$$

Let's now focus on some specific physical implementations of the Josephson effect and try to say something about the all-important constant A_1 (or equivalently I_c). For orientation let's start with a very simple phenomenological description of the effect in a thick tunnel-oxide barrier. We imagine that we may treat the Cooper pairs for this purpose as 'like' diatomic molecules, so that they are described by a Ginzburg-Landau-like wave function $\Psi(z)$ ($z =$ direction normal to barrier) which obeys a single-particle time-independent Schrödinger equation with a mass $m_p = 2m$ and a potential $V_p(z) = 2V(z)$ where $V(z)$ is the one- electron potential energy. The energy eigenvalue is taken to be 2μ in the limit $V_p \to \infty$ (no barrier penetration). All self-interactions are neglected in the barrier region. With these substitutions the problem reduces exactly to a one-particle barrier tunneling problem, in which we must minimize the energy subject to the constraint that the values of Ψ obtained at the barrier edges each correspond in magnitude to the bulk value Ψ_∞ but differ in phase by $\Delta\phi$, i.e.

$$\begin{aligned}\Psi(z=+L/2) &= \Psi_\infty e^{i\Delta\phi/2}\\ \Psi(z=-L/2) &= \Psi_\infty e^{-i\Delta\phi/2}\end{aligned} \tag{15}$$



Let $\Psi_0(z) \equiv \Psi_+$ be the under-barrier wave function for $\Delta\phi = 0$, and call the corresponding energy E_t; in the limit of weak tunneling (WKB limit) it is intuitively plausible (and true) that $\Psi_0(z)$ is simply $\Psi_L(z)+\Psi_R(z)$, where Ψ_R and Ψ_L are wave functions which reduce to Ψ_∞ at $z = \pm L/2$ respectively and fall off exponentially under the barrier. Similarly for $\Delta\varphi = \pi, \Psi(z) \equiv \Psi_-$ should be given by $\Psi_L(z) - \Psi_R(z)$ with energy E_-. For general $\Delta\varphi$ the wave function $\Psi_{\Delta\varphi}(z)$ should be given by that superposition of Ψ_+ and Ψ_- which reduces to the correct values as $z \to \pm\infty$, namely $(\cos\Delta\varphi/2)\Psi_+ + i(\sin\Delta\varphi/2)\Psi_-$; this has energy (relative to that for $\Delta\varphi = 0$)

$$E(\Delta\phi) = A(1-\cos\Delta\phi). \quad A \equiv \frac{1}{2}(E_- - E_+) \tag{16}$$

The value of $E_+ - E_-$ (or, better, of $E(\Delta\varphi)$ for $\Delta\varphi \to 0$) can be calculated from the ansatz $\Psi(\Delta\varphi : z) = \exp i\varphi(z)\cdot\Psi_0(z)$ subject to the condition $\varphi(\pm\infty) = \pm\Delta\varphi/2$ so that the only $\Delta\varphi$-dependent term is the kinetic energy, $\propto |\Psi_0(z)|^2(\nabla\varphi(z))^2$. Minimization with respect to the form of $\varphi(z)$ gives

$$A = \frac{\hbar^2}{em_p}\left[\int_{-L/2}^{L/2}\frac{dz}{\Psi_0^2(z)}\right]^{-1} \tag{17}$$

Now in the WKB limit this expression is proportional to the quantity

$$\exp - \int_{-L/2}^{L/2}\sqrt{2m_p(V_p(z)-2\mu)}\,dz.$$

Since $m_p = 2m$ and $V_p(z) = 2V(z)$, this is just the square of the single-particle matrix element t (not probability!) for tunneling through the barrier, so $A_1 \propto t^2$ as the general formulation requires.‡ Since the normal-state resistance R_N is inversely proportional to t^2, this means

$$A_1 \sim R_N^{-1} \quad (\text{or } I_c \sim R_N^{-1}) \tag{18}$$

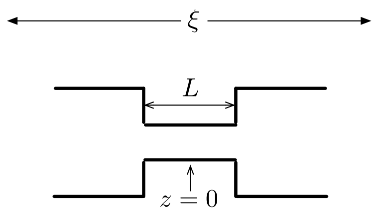
‡The quantity A_0 has, apart from the term $\sim t^2$ from (16), a large t-independent ('bulk') contribution.

We will see that this relation is more general than this specific example.

Although the above calculation is simple and suggestive, it is rather unrealistic, since it implicitly assumes not only that Cooper pairs exist in the barrier region but that their radius is small compared to the barrier width L. For a real tunnel-oxide barrier under these conditions, the constant I_c would almost certainly be unreasonably small. So let's now turn to some more realistic examples:

The first is the 'short microbridge' discussed by Tinkham (§6.2.1) and I follow his discussion with minor notational changes. The GL free energy is in dimensionless form

$$F[\Psi(z)] = \mathcal{F}_0(T) \int \Big\{ -|f|^2 + \frac{1}{2}|f|^4 + \frac{1}{2}\,\xi^2(T)\left|\frac{\partial f}{\partial z}\right|^2 \Big\}\, dz \tag{19}$$



where $f \equiv \Psi/\Psi_\infty$ (Ψ_∞ is the bulk OP), $\mathcal{F}_0(T)$ is the corresponding bulk free energy density and $\xi(T)$ is the GL healing length. If we assume a 'short' microbridge, $L \ll \xi(T)$, and assume some nonzero value of the phase difference between the two ends ($z = \pm L/2$), then the gradient term will be of order $(\xi/L)^2 \gg 1$ relative to the other two, and the resulting GL equation for the OP will just minimize it, i.e. give $\partial^2 f/\partial z^2 = 0$. This has the solution

$$\begin{aligned} f &= \frac{1}{2}\big\{e^{i\Delta\phi/2}\big(1 + z/(L/2)\big) + e^{-i\Delta\phi/2}\big(1 - z/(L/2)\big)\big\} \\ &\equiv \cos\Delta\phi/2 + i\big(z/(L/2)\big)\sin\Delta\phi/2 \end{aligned} \tag{20}$$

Note that $|f|$ is everywhere nonzero for any value of $\Delta\phi$ except $(2n+1)\pi$, for which it is zero at $z = 0$ (i.e. in the middle of the bridge). The free energy corresponding to this solution is, relative to its value for $\Delta\phi = 0$,

$$\Delta F = \frac{4\mathcal{A}}{L}\mathcal{F}_0(T)\xi^2(T)\big(\sin^2(\Delta\varphi/2)\big), \qquad \mathcal{A} = \begin{matrix}\text{cross-section area}\\ \text{of the brigde}\end{matrix} \tag{21}$$

or using the fact that $\mathcal{F}_0(T)\xi^2(T) = \frac{\hbar^2}{2m}\,\Psi_\infty^2(T)$,

$$\Delta F = \frac{\hbar^2\mathcal{A}}{mL}\,\Psi_\infty^2\big(1 - \cos\Delta\phi\big) \tag{22}$$

so that the critical current is

$$\boxed{I_c = \frac{2e\hbar}{m}\,\frac{\mathcal{A}}{L}\,\Psi_\infty^2} \tag{23}$$

As noted by Tinkham, this is again proportional to R_N^{-1}.

Note that a long microbridge ($L \gtrsim \xi(T)$) will behave rather differently: as $\Delta\phi$ is 'cranked up' the OP remains everywhere finite and several 'kinks' of 2π can develop across the ends ("hysteretic" case). For $L \gg \xi(T)$ the critical current is given by the bulk GL result, (lecture 10), $I_c = \text{const } \mathcal{A}(\alpha/\beta)\xi^{-1}(T) = \text{const } \Psi_\infty^2 \mathcal{A}/\xi$ which is much larger than the extrapolated 'short-bridge'result.

Finally, let us briefly discuss the example originally considered by Josephson, namely an ideal tunnel-oxide Hamiltonian such that the single-particle tunneling is described by the 'Bardeen-Josephson' Hamiltonian

$$\hat{H}_{\mathrm{T}} = \sum_{\mathbf{kq}\sigma} T_{\mathbf{kq}\sigma} a^\dagger_{\mathbf{k}\sigma} b_{\mathbf{q}\sigma} + \text{h.c.} \tag{24}$$

where $a_{\mathbf{k}}$, $b_{\mathbf{q}}$ describe the bulk plane-wave eigenstates in metals 1 and 2; for the moment we make no particular assumptions about the single-particle tunneling matrix element $T_{\mathbf{kq}}$ (except that it conserves spin). We note that the normal-state conductance of the junction is given (cf. lecture 8) by the simple expression

$$R_N^{-1} \equiv \frac{2\pi e^2}{\hbar} 2N_1(0)N_2(0)\overline{|T|^2} \tag{25}$$

where $\overline{|T|^2}$ means the quantity $|T_{\mathbf{kq}\sigma}|^2$ averaged over $\mathbf{k}$, $\mathbf{q}$ and σ on states near the Fermi energy.

We will confine ourselves to zero temperature and, following Josephson, evaluate the ground state energy of the coupled system by perturbation theory in $\hat{H}_{\mathrm{T}}$ up to second order. The general formula is (since $\langle 0|\hat{H}_{\mathrm{T}}|0\rangle$ is obviously zero)

$$\Delta E = \sum_n \frac{|\langle n|\hat{H}_{\mathrm{T}}|0\rangle|^2}{E_0 - E_n} \tag{26}$$

In the normal phase the contributions from the two terms in $\hat{H}_{\mathrm{T}}$ are mutually incoherent, since they give rise to different final states, and give rise to an expression proportional to the integral

$$-N_1(0)N_2(0)\,\overline{|T|^2} \int_0 \int_0 \frac{d\epsilon d\epsilon'}{\epsilon + \epsilon'} \tag{27}$$

which is convergent at the lower end but needs a physical cut-off at the upper end. In the superconducting phase we get coherence effects between the two terms in $\hat{H}_{\mathrm{T}}$, similarly to those already encountered in lecture 8: to get two Bogoliubov quasiparticles, let us say $\mathbf{k}\uparrow$ and $-\mathbf{q}\downarrow$, we can either create an electron in $\mathbf{k}\uparrow$ and annihilate one in $\mathbf{q}\uparrow$ (first term) or annihilate one in $-\mathbf{k}\downarrow$ and create one in $-\mathbf{q}\downarrow$ (second term). Making the standard Bogoliubov transformation on both the a's and the b's, and keeping only terms involving two creation operators (all others vanish when applied to the ground state), we obtain an expression for the relevant matrix element which is a special case of the formula of lecture 8

$$M_{\mathbf{kq}\sigma} = T_{\mathbf{kq}\sigma}(u_{\mathbf{k}}v_{\mathbf{q}} + \eta v_{\mathbf{k}}u_{\mathbf{q}}) \tag{28}$$

where η describes the behavior of the single-particle matrix element under time reversal, i.e. $T_{-\mathbf{k},-\mathbf{q},-\sigma} = \eta T_{\mathbf{kq}\sigma}$. Except in very special cases involving e.g., magnetic impurities in the junction, T is almost invariably pure even under time reversal, i.e., $\eta = +1$.

Inserting this value and noting that for the process considered $E_n - E_0 = E_{\mathbf{k}} + E_{\mathbf{q}}$, we find

$$\begin{aligned}\Delta E &= -\sum_{\mathbf{kq}\sigma} |T_{\mathbf{kq}\sigma}|^2 \, \frac{\left(u_{\mathbf{k}}^2 |v_{\mathbf{q}}|^2 + |v_{\mathbf{k}}|^2 u_{\mathbf{q}}^2\right) + 2\mathrm{Re}\left\{ \left(u_{\mathbf{k}} v_{\mathbf{k}}\right)^* \left(u_{\mathbf{q}} v_{\mathbf{q}}\right) \right\}}{E_{\mathbf{k}} + E_{\mathbf{q}}} = \\ &= -N_1(0) N_2(0) \overline{|T|^2} \int_0^\infty \!\! \int_0^\infty d\epsilon d\epsilon' \, \frac{\left(1 - (\epsilon\epsilon'/EE')\right) + 2\mathrm{Re}\left(\Delta_{\mathbf{k}}^* \Delta_{\mathbf{q}}/EE'\right)}{E + E'}\end{aligned} \tag{29}$$

The first term depends only on the modulus of $\Delta_{\mathbf{k}}$ and $\Delta_{\mathbf{q}}$. However, the second depends on the *relative phase* across the junction: $\mathrm{Re}\left(\Delta_{\mathbf{k}}^* \Delta_{\mathbf{q}}/EE'\right) \equiv |\Delta_L||\Delta_R| \cos \Delta\phi$.This might at first sight seem surprising, since we have always emphasized that the *absolute* phase of the $v_{\mathbf{k}}$'s (hence of Δ) has no physical significance. However, the *difference* between the (common) phase ϕ_L of the $v_{\mathbf{k}}$'s (referring to the L superconductor) and the (common) phase ϕ_R of the $v_{\mathbf{q}}$'s referring to the R one does have a physical meaning. The reason is that the ground state wave function *cannot* be written as a product of wave functions which conserve N_L and N_R separately, i.e. of the form

$$\Psi = \Psi_{N_L}^L \Psi_{N_R}^R \equiv \int d\phi_L \, \Psi_L(\phi_L) \exp -iN_L\phi_L \cdot \int d\phi_R \, \Psi_R(\phi_R) \exp -iN_R\phi_R \tag{30}$$

(wrong)

rather it is of the form

$$\Psi = \int d\phi_{\mathrm{tot}} \, \Psi_L(\phi_L) \Psi_R(\phi_R) \exp -iN\phi_{\mathrm{tot}} \tag{31}$$

where N is the total number of electrons and $\phi_{\mathrm{tot}} = (\phi_R + \phi_L)/2$ is the 'average' phase; thus N is conserved but the difference $N_L - N_R$ is not.

It is convenient to add and subtract, in the integrand of the expression for ΔE, a term $|\Delta|^2/EE'$ (where for simplicity we now assume that the superconductors are identical); then, remembering to subtract off a term corresponding to the normal limit ($\Delta \to 0$, $E \to |\epsilon|$ etc)

$$\begin{aligned}\Delta E &= \Delta E^{(1)} + \Delta E^{(2)} \\ \Delta E^{(1)} &\equiv -N_1(0) N_2(0) \overline{|T|^2} \int_0^\infty \!\! \int_0^\infty d\epsilon d\epsilon' \, \frac{\left(1 - (\epsilon\epsilon' - \Delta^2)/EE'\right)}{E + E'} - (\Delta \to 0) \\ \Delta E^{(2)} &\equiv +N_1(0) N_2(0) \overline{|T|^2} \int_0^\infty \!\! \int_0^\infty d\epsilon d\epsilon' \, \frac{|\Delta|^2}{(E + E')EE'} \, (1 - \cos \Delta\phi)\end{aligned} \tag{32}$$

The first term turns out to vanish (and is in any case independent of $\Delta\varphi$), while the coefficient in the second is clearly of order $\Delta \, N_1(0) N_2(0) \overline{|T|^2}$, i.e. proportional to Δ/R_N. On working out the numerical constants, we get the famous result

$$\boxed{I_c = \pi\Delta/eR_N} \quad (T = 0) \tag{33}$$

Most tunnel oxide junctions between 'classic' superconductors seem to satisfy the relation reasonably well.

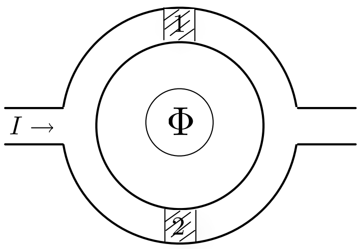
The finite-temperature generalization is reasonably straightforward and yields the result

$$I_c(T) = \frac{\pi\Delta(T)}{eR_N} \tanh \Delta(T)/2T \tag{34}$$

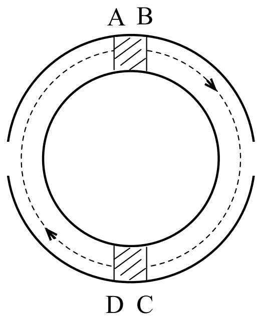
so that the critical current vanishes as $\Delta^2(T)$ (or $\Psi^2(T)$) as $T \to T_c$.

In all the examples considered so far, the critical current I_c has been proportional to the normal-state conductance R_N^{-1} of the weak limit. However, this result is not universal: in particular, it does not hold for SNS junctions, for which R_N^{-1} scales linearly with the inverse thickness d^{-1} while I_c decreases exponentially with d.

In all the above examples, we implicitly assumed that while the phase of the two condensates is different on each side it is constant across the cross-section of the junction. However, this need not be the case, in particular if a magnetic field is applied. To introduce this subject, let's start with the setup known as a dc SQUID:



Assume for simplicity that the two junctions are identical and that the flux is applied under 'Aharonov-Bohm' conditions, i.e. so that no magnetic field penetrates the metal itself. Then the current flowing through arm 1 (assuming it is less than I_c) is $I_c \sin \Delta\phi_1$ and that through arm 2 is $I_c \sin \Delta\phi_2$. However, $\Delta\phi_1$ and $\Delta\phi_2$ *are not independent*: if we consider a path going once around the ring well inside the London penetration depth, we find that since



$$\Delta\phi_{AD} = \frac{e}{\hbar}\int_D^A \mathbf{A} d\boldsymbol{l}, \quad \Delta\phi_{CB} = \frac{e}{\hbar}\int_B^C \mathbf{A} d\boldsymbol{l} \tag{35}$$

and the integral of $\mathbf{A}$ across the junctions themselves is negligible, the total phase difference around the ring, $(A \to A)$ is $\Delta\phi_1 - \Delta\phi_2 - 2n\Phi/\Phi_0$. This total phase difference must be zero modulo 2π, and thus we conclude

$$\boxed{\Delta\phi_1 - \Delta\phi_2 = 2\pi\Phi/\Phi_0} \tag{36}$$

($\Delta\phi_1$ and $\Delta\phi_2$ both taken 'left to right')

Thus the total current I through the device is ($\Delta\phi_1 \equiv \xi + \pi\Phi/\Phi_0$, $\Delta\phi_2 \equiv \xi - \pi\Phi/\Phi_0$)

$$I = I_c(\sin\Delta\phi_1 + \sin\Delta\phi_2) = 2I_c \sin\xi \cos\pi\Phi/\Phi_0 = 2I_c(\Phi)\sin\xi \tag{37}$$

so that the total critical current of the device is (since the sign of I_c is irrelevant)

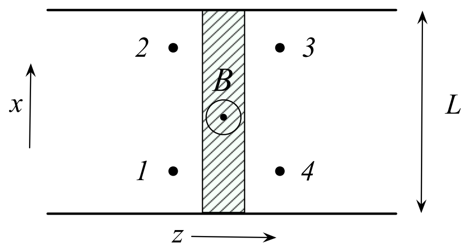
$$I_c(\Phi) \equiv 2I_c|\cos\pi\Phi/\Phi_0| \tag{38}$$

This has a maximum ($2I_c$) when $\Phi = n\Phi_0$ (n integral) and a minimum (zero) when $\Phi = (n + 1/2)\Phi_0$.

A single junction in a transverse dc magnetic field may be regarded as the continuum version of the above dc SQUID. We assume that the current density at each value of the transverse coordinate x, $\tilde{I}(x)$, is equal to $\tilde{I}_c \sin \Delta\phi(x)$ where $\tilde{I}_c$ is the critical current per unit length. Suppose, moreover, that no current flows along the junction in the bulk (i.e., at $z \gg \lambda$, see below). Then

$$\Delta\phi(x_1) - \Delta\phi(x_2) = \int_1^2 \frac{e}{\hbar}\mathbf{A}d\boldsymbol{l} - \int_3^4 \frac{e}{\hbar}\mathbf{A}d\boldsymbol{l} \quad \Big[\equiv \frac{e}{\hbar}\Big\{\int_2^1 + \int_3^4\Big\}\mathbf{A}d\boldsymbol{l}\Big] \tag{39}$$

If the field B in the junction is constant, we can choose $\big(A_{x,L} - A_{x,R}\big)(x) = Bd_{eff}$, where d_{eff} is the effective "width" of the junction (see below), and then the integral is $(eBd_{\text{eff}}/\hbar)(x_1 - x_2)$, i.e. we can write



$$\Delta\phi(x) = \Delta\phi + (eBd_{\text{eff}}/\hbar)x \tag{40}$$

where x is measured (say) from the middle of the junction. The total current flowing along the junction is therefore

$$I = \int \tilde{I}(x)\,dx = \tilde{I}_c \int_{L/2}^{-L/2} \sin\big(\Delta\phi + (eBd_{eff}/\hbar)x\big)\,dx \tag{41}$$

When we perform the integration over x, the term proportional to $\cos\Delta\phi$ vanishes by symmetry, so using $I_c \equiv \tilde{I}_c L$ we get

$$I = I_c(\Phi)\sin\Delta\phi \tag{42}$$

$$I_c(\Phi) = \frac{2\tilde{I}_c}{eBd_{\text{eff}}/\hbar}\sin\big(eBd_{\text{eff}}L/2\hbar\big) \equiv \frac{I_c \sin \pi\Phi/\Phi_0}{\pi\Phi/\Phi_0} \tag{43}$$

where $\Phi \equiv BdL$ is the total flux through the junction (cf. below). The sign of the critical current is irrelevant, so its magnitude is $|\sin(\pi\Phi/\Phi_0)|/(\pi\Phi/\Phi_0)$, giving a 'Fraunhofer diffraction pattern' as a function of Φ.

There is one delicate point about the above argument. What, exactly, should we take as the 'effective junction width' d_{eff}? Strictly speaking, our assumption about 'no transverse current flow in bulk' will be true only if 'bulk' means 'at distances from the junction $\gg \lambda$'. Thus, we would guess intuitively that d_{eff} would be something like $d + 2\lambda$ where d is the true (geometrical) width. This guess is in fact correct, but to prove it we should have to do a more detailed calculation of the current flow pattern in the region near the junction. The result provides one of the most accurate ways of measuring the penetration depth $\lambda(T)$: note that even if we do not know the value of d accurately, differences $\Delta\lambda(T) \equiv \lambda(T) - \lambda(0)$ do not depend on this.

[Josephson penetration depth¶ $\lambda_{\text{J}} = (\Phi_0/4\pi\mu_0 I_c d_{\text{eff}})^{1/2} \sim$ (typically) ~ 1 mm.]

¶ $J \sim \frac{e\hbar}{2m}\rho_s\nabla\phi \to I_c d_{\text{eff}}\nabla\phi \Rightarrow \rho^s_{\text{eff}} = 2mI_c d_{\text{eff}}/\hbar e$ in London equation.

Finally we turn briefly to the question of the dynamics of the junction. Let us consider the standard case, where the external bias current I_{ext} is fixed, but we will not assume a priori that the phase $\Delta\phi$ is constant in time. What this means is that since the current though the junction itself is $I_c \sin\Delta\phi(t)$, and I_{ext} is fixed, if $\Delta\phi$ varies in time there must be a charge build-up $Q(t)$ in the electrodes. For a finite (i.e., non-infinite) capacitance this will give rise to a voltage difference, which will then by the second Josephson equation drive the phase rotation. Furthermore, if such a voltage develops we might expect there to be a 'normal' current driven through the junction in addition to the supercurrent: let us describe this by a phenomenological ohmic resistance R_n (which in general may be a strong function of T). The relevant equations are then, if $I(t)$ is the total current actually crossing the junction

$$\begin{gathered} I(t) = I_c \sin\Delta\phi(t) + V(t)/R_n = I_{\text{ext}} - \dot{Q}(t) \\ Q(t) = CV(t) \\ \Delta\dot{\phi} = (2e/\hbar)V(t) \end{gathered} \tag{44}$$

which when combined yield a second-order differential equation for $\Delta\phi(t)$:

$$\boxed{C\Delta\ddot{\phi}(t) + \frac{\Delta\dot{\phi}}{R_n} + \frac{\Phi_0 I_c}{2\pi}\sin\Delta\phi = \frac{\Phi_0 I_{\text{ext}}}{2\pi}} \tag{45}$$

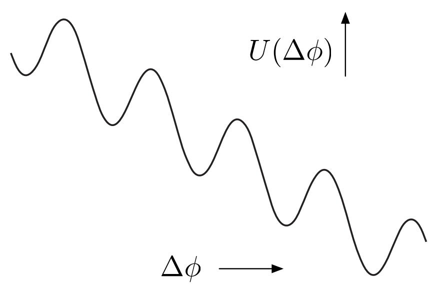
This is the so-called RSJC model ('resistively shunted junction with capacitance'). In the limit of zero dissipation ($R_n \to \infty$) and small oscillations of $\Delta\varphi$ around equilibrium this yields a simple harmonic oscillation of $\Delta\phi$ (Josephson plasma resonance): for $I_{\text{ext}} = 0$ the frequency is

$$\omega_{\text{J}} = \left(\frac{\Phi_0 I_c}{2\pi C}\right)^{1/2} \tag{46}$$

As I_{ext} is increased, the resonance frequency decreases and $\to 0$ as $I_{\text{ext}} \to I_c$.

The above equation, which is classical, may be regarded as the equation of motion of a 'particle' of 'mass' C and friction coefficient R_n^{-1} moving in the so-called 'washboard potential'

$$U(\Delta\varphi) = \frac{\Phi_0}{2\pi}\left(I_c \sin\Delta\phi - I_{\text{ext}}\Delta\phi\right) \tag{47}$$



(However, some caution is required when attempting to discuss quantum-mechanical effects in this analogy.) For various applications, see Tinkham sections 6.3–7 and chapter 7.