Superconducting alloys: electromagnetic response of clean and dirty superconductors

*References AJL QL 5.8. [Tinkham 3.10-11, de Gennes ch. 5]**

Experimental fact: superconductivity fairly insensitive to nonmagnetic impurities, but rapidly destroyed by *magnetic* ones (i.e., those corresponding to local moments). Why? Consider in this lecture the case of *spinless*§ impurity potential, so

$$\hat{H} = \hat{H}_0 + \frac{1}{2}\sum_i V(\mathbf{r}_i - \mathbf{r}_j), \quad \hat{H}_0 = \sum_i \frac{\mathbf{p}_i^2}{2m} + U(\mathbf{r}_i) \tag{1}$$

$U(\mathbf{r}_i)$ includes 'random' potential due to impurities (spin-independent and T-invariant).

Consider eigenstates of one-particle term $\hat{H}_0$. Since spin conserved, these are of form $|\sigma\rangle\phi_n(\mathbf{r})$ with $\epsilon_n \equiv \xi_n - \mu$ independent of $|\sigma\rangle$ $(= |\uparrow\rangle, |\downarrow\rangle)$. If $\phi_n(\mathbf{r})$ is eigenstate, then since by T-invariance $\hat{H}^* = \hat{H}$ so is $\phi_n^* \equiv \phi_{\bar{n}}$, which may or may not be different from ϕ_n (doesn't matter!). [In translation-invariant case, can if we like choose instead of $\exp \pm i\mathbf{kr}$ the real functions $\cos \mathbf{kr}$, $\sin \mathbf{kr}$.] Moreover, crucially, $\phi_{\overline{n}}$ has exactly the same energy eigenvalue ϵ_n as ϕ.

Anderson prescription: *pair in time-reversed states* $\phi_n|\uparrow\rangle$, $\phi_{\bar{n}}|\downarrow\rangle$, i.e. if $a^\dagger_{n\uparrow}$, $a^\dagger_{\bar{n}\downarrow}$ create these states,

$$\Psi_{\mathrm{BCS}} = \prod_n (u_n + v_n a^\dagger_{n\uparrow} a^\dagger_{\bar{n}\downarrow})|\mathrm{vac}\rangle \tag{2}$$

Everything now goes through exactly as in translation-invariant case, though some formulae have to be generalized. In particular, expression for contribution to potential energy from pairing is

$$\langle V\rangle_{\mathrm{pair}} = \iint V(\mathbf{r}-\mathbf{r}')|F(\mathbf{r},\mathbf{r}')|^2\, d\mathbf{r}d\mathbf{r}' \tag{3}$$

where $F(\mathbf{r},\mathbf{r}')$ is defined by

$$F(\mathbf{r},\mathbf{r}') \equiv \langle \psi^\dagger_\uparrow(\mathbf{r})\psi^\dagger_\downarrow(\mathbf{r}')\rangle \tag{4}$$

and is now in general not a function simply of $\mathbf{r}-\mathbf{r}'$ as in the translation-invariant case. If we define $F_n \equiv u_n v_n$, then

$$F(\mathbf{r},\mathbf{r}') = \sum_n F_n \phi_n(\mathbf{r})\phi_{\bar{n}}(\mathbf{r}') \tag{5}$$

Now we are generally interested in the value of $F(\mathbf{r},\mathbf{r}')$ at short distances (typically $\sim k_{\mathrm{D}}^{-1}$) since it is only there that V is appreciable. Thus, we consider $F(\mathbf{r},\mathbf{r})$. Since $\phi_{\bar{n}}(\mathbf{r}) \equiv \phi_n^*(\mathbf{r})$, we get

$$F(\mathbf{r},\mathbf{r}) = \sum_n F_n |\phi_n(\mathbf{r})|^2 \tag{6}$$

*Tinkham goes into more detail than needed here. de Gennes is at about the right level but uses the BdG equations, which we will get to in lecture 11.

§Magnetic impurities will be considered in lecture 12.

The crucial point is that the quantity multiplying F_n is positive definite and its average value is not much different from that of a plane wave. Thus the PE can be comparable to what it was in the absence of the impurities. In fact, we have approximately

$$\langle V \rangle \approx V_0 \sum_{nn'} F_n F^*_{n'} = V_0 \sum_{nn'} \frac{\Delta_n}{2E_n} \frac{\Delta_{n'}}{2E_{n'}} \tag{7}$$

Since the one-particle energy is just the generalization of that for the translation-invariant case, i.e.

$$\langle H_o \rangle = \sum_n (1 - \epsilon_n/E_n)\epsilon_n \tag{8}$$

the gap equation similarly comes out as

$$\Delta_m = -V_0 \sum_n \frac{\Delta_n}{2E_n} \tanh \beta E_n/2 \tag{9}$$

As in the translation-invariant case it is clear that a possible solution is $\Delta_m = \text{const} = \Delta$, and the equation then reduces to

$$1 = -V_0 \int \left(\frac{dn}{d\epsilon}\right) d\epsilon \, \frac{\tanh \beta E/2}{E}, \quad (E \equiv \sqrt{\epsilon^2 + \Delta^2}) \tag{10}$$

The only difference with the translation-invariant case is that the DOS $dn/d\epsilon \equiv \sum_n \delta(\epsilon - \epsilon_n)$ is that in the presence of the impurities. But unless we are in or very close to the limit of localization (which requires a much larger impurity concentration than normally considered) the average number of states over an energy range $\ll \Delta$ is very little affected, so we can take $dn/d\epsilon$ equal to the 'pure' value $2N(0)$ and take it out of the integral. It is then clear that (in zero magnetic field) the thermodynamics is *identical* to that of the pure system: in particular T_c should be very insensitive to nonmagnetic alloying.

Generalization: There are some cases where the impurity Hamiltonian is spin-dependent but nevertheless preserves time-reversal invariance (example: spin-orbit scattering in heavy metals). In this case, we can still pair in T-reversed states (though they are in general not eigenfunctions of spin) and the same conclusions go through. For the case of non- T-invariant perturbations, see lecture 12.

Behavior of the pair wave function

In general, if we write $\mathbf{R} \equiv (\mathbf{r}_1 + \mathbf{r}_2)/2$, $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ we have

$$F(\mathbf{r}_1, \mathbf{r}_2) \equiv F(\mathbf{R}, \mathbf{r}) = \sum_n \left\{ \frac{\Delta_n}{2E_n} \tanh \beta E_n/2 \right\} \phi^*_n(\mathbf{R} + \mathbf{r}/2)\phi_n(\mathbf{R} - \mathbf{r}/2) \tag{11}$$

As we have seen, for $\mathbf{r} = 0$ this quantity is not strongly dependent on $\mathbf{R}$ and is close to its value for the pure metal. For $\mathbf{r} \neq 0$ it is also unlikely to be strongly dependent on $\mathbf{R}$, so we can imagine averaging over $\mathbf{R}$ and writing a quantity

$$\overline{F(\mathbf{r})} = \sum_n \left\{ \frac{\Delta_n}{2E_n} \tanh \beta E_n/2 \right\} \overline{\phi^*_n(\mathbf{R} + \mathbf{r}/2)\phi_n(\mathbf{R} - \mathbf{r}/2)} \tag{12}$$

The quantity $\int \mathbf{r}^2 |\overline{F(\mathbf{r})}|^2 \, d\mathbf{r} / \int |\overline{F(\mathbf{r})}|^2 \, d\mathbf{r}$ can be taken as the mean-square radius of a Cooper pair in the dirty metal. In the limit of no impurities the definition (12) reduces to our earlier expression

$$F(\mathbf{r}) = \int d\mathbf{k} \, \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}} \, \tanh \beta E_{\mathbf{k}}/2 \cdot \exp i\mathbf{k}\mathbf{r} \tag{13}$$

and we saw this had a rms radius $\sim \hbar v_{\mathrm{F}}/\Delta \equiv \mathrm{const}\, \xi_0$. What about the 'dirty limit' (defined by mfp $l \ll \xi_0$)?

Intuitive argument: $F(\mathbf{r})$ will drop below its $\mathbf{r} = 0$ value as soon as the *difference* in the phase difference between $\phi_n(\mathbf{R} + \mathbf{r}/2)$ and $\phi_n(\mathbf{R} - \mathbf{r}/2)$ for different n which are well represented in the sum, becomes $\sim 2\pi$. We can calculate this by a semiclassical argument: a wave packet with spread in energy ΔE will be dephased by $\sim 2\pi$ in a time $\Delta t \sim h/\Delta E$ later. In this case, $\Delta E \sim \Delta$ (or $k_{\mathrm{B}}T_c$), so $\Delta t \sim h/\Delta$. Now, in a clean metal the distance which the packet has travelled by time Δt is the 'ballistic' time $r \sim v_{\mathrm{F}}\Delta t$, so we find that dephasing is more or less complete at $r \sim \hbar v_{\mathrm{F}}/\Delta \sim \xi_0$ in agreement with the exact result. For a dirty metal, on the other hand, the behavior is diffusive and we have $r^2 \sim D\Delta t$, where the diffusion coefficient D is $(1/3)v_{\mathrm{F}}l$ (l = (elastic) mfp). Hence, putting $\Delta t \sim \hbar/\Delta$ as above, we find for the pair mean-square radius

$$r_{\mathrm{p}}^2 \sim D\hbar/\Delta \sim (\hbar v_{\mathrm{F}}/\Delta) l \tag{14}$$

or in other words

$$\boxed{r_{\mathrm{p}} \sim (\xi_0 l)^{1/2}} \tag{15}$$

Thus the 'radius' of a Cooper pair is reduced in the dirty limit by a factor $(l/\xi_0)^{1/2}$.

Electromagnetic response of clean and dirty superconductors†

Confine ourselves to *linear* properties (in external field). Then, most generally, can express EM response in terms of the reaction of the response of the electric current $J_\mu(\mathbf{r}, t)$ to the applied EM vector potential $A_\nu(\mathbf{r}', t')$:‡ since system invariant under time translation, we have

$$K_{\mu\nu}(\mathbf{r}, \mathbf{r}'; t, t') \equiv \delta J_\mu(\mathbf{r}, t)/\delta A_\nu(\mathbf{r}', t') = K_{\mu\nu}(\mathbf{r}, \mathbf{r}'; t - t') \tag{16}$$

and can take the Fourier transform $K_{\mu\nu}(\mathbf{r}, \mathbf{r}'; \omega)$ with respect $t - t'$. Convenient for the moment *not* to take spatial FT. Recall that $\boldsymbol{E}$ and $\mathbf{B}$ related to $\mathbf{A}$ by

$$\boldsymbol{E} = -\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \mathrm{curl}\, \mathbf{A} \tag{17}$$

A *static longitudinal* vector potential (i.e. one such that $\mathbf{A} = \mathbf{A}(\mathbf{r})$, $\mathrm{curl}\, \mathbf{A} = 0$ can always be gauged away ($\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi$) and thus produces no physical effects. We work in the appropriate (Landau) gauge, $\mathrm{div}\, \mathbf{A}(\mathbf{r}, t) = 0$: $\mathbf{A}$ is 'purely transverse'.

†Refs.: de Gennes pp. 137–143, 157–182, 195–6, 210–227; Schrieffer, sections 8.1–4; Tinkham, section 2.10

‡Since we can always choose gauge so that EM scalar potential $\phi(\mathbf{r}, t) \equiv 0$

To calculate the response,we need to bear in mind that in the presence of a magnetic vector potential the EM current $\mathbf{J}(\mathbf{r},t)$ has the form (apart from symmetrization)

$$\mathbf{J}(\mathbf{r},t)=\sum_i \delta\big(\mathbf{r}-\mathbf{r}_i(t)\big)\frac{e}{m}\left(\mathbf{p}_i-e\mathbf{A}(\mathbf{r}_i,t)\right) \tag{18}$$

Therefore, there is an explicit 'diamagnetic' term $-\frac{e^2}{m}\sum_i \delta(\mathbf{r}-\mathbf{r}_i)\mathbf{A}(\mathbf{r}_i,t)\equiv -n(\mathbf{r},t)\frac{e^2}{m}\mathbf{A}(\mathbf{r},t)$ where $n(\mathbf{r},t)$ is the density of electrons. It is usually an excellent approximation to set in this term $n(\mathbf{r},t)\equiv n$, the equilibrium density. Now, the perturbation terms in the Hamiltonian in the presence of a vector potential $\mathbf{A}(\mathbf{r},t)$ are of the form

$$-\int \tilde{\mathbf{J}}(\mathbf{r},t)\mathbf{A}(\mathbf{r},t)\,d\mathbf{r}+\frac{1}{2}\int n(\mathbf{r},t)\mathbf{A}^2(\mathbf{r},t)\,d\mathbf{r} \tag{19}$$

where we define for convenience (up to symmetrization)

$$\tilde{\mathbf{J}}(\mathbf{r},t)\equiv\sum_i \delta\big(\mathbf{r}-\mathbf{r}_i(t)\big)\frac{e}{m}\,\mathbf{p}_i \tag{20}$$

The second term in (19) does not contribute to linear order, so we can write

$$K_{\mu\nu}(\mathbf{r},\mathbf{r}';t-t')=\tilde{K}_{\mu\nu}(\mathbf{r},\mathbf{r}';t-t')-\frac{ne^2}{m}\delta(\mathbf{r}-\mathbf{r}') \tag{21}$$

where $\tilde{K}_{\mu\nu}(\mathbf{r},\mathbf{r}';t-t')$ is the standard response function $\delta J_\mu(\mathbf{r},t)/\delta A_\nu(\mathbf{r}',t')|_{\text{no diamagnetic term}}$, i.e. the response of $\tilde{\mathbf{J}}(\mathbf{r},t)$ to a field which couples linearly to it, and we took $n(\mathbf{r},t)\approx$ const $\equiv n$. From standard response function theory, the time Fourier transform of $\tilde{K}_{\mu\nu}(\mathbf{r},\mathbf{r}';t-t')$ is explicitly

$$\tilde{K}_{\mu\nu}(\mathbf{r},\mathbf{r}';\omega)=Z^{-1}\sum_m e^{-\beta\epsilon_m}\sum_n \frac{\langle m|\tilde{J}_\mu(\mathbf{r})|n\rangle\langle n|\tilde{J}_\nu(\mathbf{r}')|m\rangle}{\epsilon_n-\epsilon_m+\hbar\omega+i\delta}+\frac{\langle m|\tilde{J}_\mu(\mathbf{r})|n\rangle\langle n|\tilde{J}_\nu(\mathbf{r}')|m\rangle}{\epsilon_n-\epsilon_m-\hbar\omega-i\delta} \tag{22}$$

where the ϵ_m are the *many-body* excitation energies of the system. The problem reduces to the calculation of the expression on the RHS of (22).

Consider first the longitudinal case. We know that in this case, whatever the state of the system, the total response to a *static* field, $K_{\mu\nu}(\mathbf{r},\mathbf{r}':0)\equiv\tilde{K}_{\mu\nu}(\mathbf{r},\mathbf{r}';0)-\frac{ne^2}{m}\delta(\mathbf{r}-\mathbf{r}')$, must vanish, and this in fact turns out to be guaranteed by the f-sum rule.

Consider next the case of a normal system, i.e. one lacking LRO and again consider the static limit ($\omega=0$). In this case, if the quantity $\mathbf{A}(\mathbf{r})$ is sufficiently slowly varying in space, the system should not 'know the difference' between longitudinal and transverse: so we expect that in this limit the result should be the same, i.e. $\delta\mathbf{J}(\mathbf{r})/\delta\mathbf{A}(\mathbf{r}')=0$ (no Meissner effect). Explicit evaluation of the matrix elements for the normal Fermi system confirms this. (For finite values of $\mathbf{q}$, where $K_{\mu\nu(q)}$ is the Fourier transform of $K_{\mu\nu(r,r')}$ with respect to $(r-r')$, we get a small difference between L and T corresponding to Landau diamagnetism). In the superconducting case, however, the LRO permits the system to 'know the difference', and we have no a priori guarantee that the cancellation

will occur even for $\omega = 0$. The reason is that although the quasiparticles (normal component) do not 'know the difference', the condensate does: it can flow only longitudinally, however long the 'wavelength' of the disturbance!

To calculate the transverse $\tilde{K}_{\mu\nu}$ we can neglect the condensate and thus proceed as in the translation invariant case by making the Bogoliubov transformation

$$a^\dagger_{n\sigma} = u_n \alpha^\dagger_{n\sigma} + \sigma v_n \alpha_{\bar{n},-\sigma} \tag{23}$$

etc. ($u_{\bar{n}} \equiv u_n$, $v_{\bar{n}} \equiv v_n$) Since the matrix element of the current between normal eigenstates m, n is

$$\tilde{\mathbf{J}}_{mn}(\mathbf{r}) \equiv \langle m|\tilde{\mathbf{J}}(\mathbf{r})|n\rangle = -i\hbar\Big(\phi^*_m(\mathbf{r})\boldsymbol{\nabla}\phi_n(\mathbf{r}) - \phi_n(\mathbf{r})\boldsymbol{\nabla}\phi^*_m(\mathbf{r})\Big) \tag{24}$$

it is *odd* under T-reversal, i.e. matrix elements are type II. Hence the effect (cf. lecture 8) is to multiply (e.g.) the '2-particle creation' terms by a factor

$$\frac{1}{2}\,\frac{E_m E_n - \epsilon_m \epsilon_n - \Delta^2}{E_m E_n} \tag{25}$$

as well as replacing $\epsilon_m \to E_m$ in the denominators and thermal factors. The resulting expression is messy in general, so let's specialize at this point to the case $\omega = 0$, $T = 0$ (static zero temperature response):

$$\tilde{K}_{\mu\nu}(\mathbf{r},\mathbf{r}';0)_{T=0} = \sum_{mn} \frac{1}{2}\,\frac{E_m E_n - \epsilon_m \epsilon_n - \Delta^2}{E_m E_n (E_m + E_n)} \big\{ J^{(\mu)}_{mn}(\mathbf{r}) J^{(\nu)*}_{mn}(\mathbf{r}') + \text{c.c.}\big\} \tag{26}$$

At first sight, one cannot get much further without an explicit knowledge of the matrix elements $\tilde{J}_{mn}(\mathbf{r})$, which in turn would require a knowledge of the exact normal eigenstates $\phi_m(\mathbf{r})$. However, an important observation is that the sum (26) is very similar to that occurring in the normal-state conductivity $\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}';\omega)$. In fact, it is easy to establish that within the simple noninteracting electron gas picture, the latter is given by

$$\mathrm{Re}\,\sigma_{\mu\nu}(\omega) = \omega^{-1} \sum_{mn} \big\{ \tilde{J}^{(\mu)}_{mn}(\mathbf{r}) \tilde{J}^{(\nu)*}_{mn}(\mathbf{r}') + \text{c.c.}\big\}\,(f_m - f_n)\,\frac{\pi}{2}\,\delta(\epsilon_m - \epsilon_n - \hbar\omega) \tag{27}$$

($f_m \equiv$ Fermi function). Let's take the limit $k_\mathrm{B}T \gg \hbar\omega$, where $f_m - f_n$ can be approximated by $\hbar\omega\,\partial f/\partial\epsilon$. Let us define a quantity (actually a function of $\mathbf{r}$ and $\mathbf{r}'$)

$$Q_{\mu\nu}(\epsilon,\epsilon') \equiv \sum_{mn} \big\{ \tilde{J}^{(\mu)}_{mn}(\mathbf{r}) \tilde{J}^{(\nu)*}_{mn}(\mathbf{r}') + \text{c.c.}\big\} \delta(\epsilon_m - \epsilon)\delta(\epsilon_n - \epsilon') \tag{28}$$

The point of doing this is that $Q_{\mu\nu}(\epsilon,\epsilon')$ is only weakly a function of $(\epsilon+\epsilon')/2$, and thus can approximated by $Q_{\mu\nu}(\epsilon - \epsilon';\mathbf{r},\mathbf{r}')$. Then we can write

$$\tilde{K}_{\mu\nu}(\mathbf{r},\mathbf{r}';0)_{T=0} = \iint d\epsilon d\epsilon' \,\frac{1}{2}\,\frac{EE' - \epsilon\epsilon' - \Delta^2}{EE'(E+E')} Q_{\mu\nu}(\epsilon - \epsilon';\mathbf{r},\mathbf{r}') \tag{29}$$

On the other hand, the above formula for $\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}';\omega)$ becomes for $k_{\rm B}T \gg \hbar\omega$

$$\begin{aligned}\mathrm{Re}\,\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}';\omega) &= \frac{\pi}{2}\iint d\epsilon d\epsilon' \frac{\partial f}{\partial \epsilon}\,\delta(\epsilon-\epsilon'-\hbar\omega)Q(\epsilon-\epsilon';\mathbf{r},\mathbf{r}') \\ &= \frac{\pi}{2}\,\hbar\int d\epsilon \frac{\partial f}{\partial\epsilon}\,Q_{\mu\nu}(\hbar\omega) = \frac{\pi}{2}Q_{\mu\nu}(\hbar\omega;\mathbf{r},\mathbf{r}')\end{aligned} \tag{30}$$

Finally, substituting (30) (backwards) into the formula for $\tilde{K}$ and adding the diamagnetic term, we get

$$\boxed{K_{\mu\nu}(\mathbf{r},\mathbf{r}';0)_{T=0} = \frac{2}{\pi}\iint d\epsilon d\epsilon' \frac{1}{2}\frac{EE'-\epsilon\epsilon'-\Delta^2}{EE'(E+E')}\,\mathrm{Re}\,\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}';\epsilon-\epsilon') - \frac{ne^2}{m}\delta_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}')} \tag{31}$$

Thus, we have succeeded in expressing the 'Meissner kernel' $K_{\mu\nu}(\mathbf{r},\mathbf{r}';0)$ as a function only of the gap Δ and the nonlocal conductivity $\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}';\omega)$ of the normal phase (for $k_{\rm B}T \gg \hbar\omega$).

For diffuse scattering and $\mathbf{J}$ and $\boldsymbol{E}$ parallel to the surface, the Chambers formula for $\sigma_{\mu\nu}$ is

$$\sigma_{\mu\nu}(\mathbf{r},\mathbf{r}';\omega) = \frac{e^2}{2\pi}v_{\rm F}\,\frac{1}{2}\left(\frac{dn}{d\epsilon}\right)\frac{R_\mu R_\nu}{R^4}\,\exp i\omega R/v_{\rm F}\,\exp -R/l \tag{32}$$

(l = mean free path) where $R \equiv |\mathbf{r}-\mathbf{r}'|$, This can be inserted in the boxed equation above and the energy integrals calculated: the final result is very close to the formula originally guessed by Pippard on phenomenological grounds, namely

$$K_{\mu\nu}(\mathbf{r},\mathbf{r}';0) \equiv \frac{\delta J_\mu(\mathbf{r})}{\delta A_\nu(\mathbf{r}')} \approx -\frac{3ne^2}{4\pi m\xi_0}\,\frac{R_\mu R_\nu}{R^4}\,\exp -R(1/l+1/\xi_0) \tag{33}$$

Thus, for a clean metal ($l \gg \xi_0$), the 'nonlocality length' of the EM response is the pair radius ξ_0 (up to a numerical factor). For a dirty metal, however, this length is *not* the pair radius $(\xi_0 l)^{1/2}$ but rather the smaller mean free path l. Thus, if we *define* the (zero-T) superfluid density ρ_s by the integral of $-me^2K_{\mu\nu}(\mathbf{r},\mathbf{r}';0)$ over $\mathbf{r}'$ at $T=0$, it is ρ for a pure metal and $(l/\xi_0)\rho$ for a dirty one.

Meissner Effect: Penetration Depth

If the self-consistently calculated penetration depth λ turns out to be much larger than the quantity $(1/l+1/\xi_0)^{-1}$ ($\sim \min(l,\xi_0)$) then we can write

$$J_\mu(\mathbf{r}) = \int K_{\mu\nu}(\mathbf{r},\mathbf{r}';0)A_\nu(\mathbf{r}')\,d\mathbf{r}' \approx \Big\{\int K_{\mu\nu}(\mathbf{r},\mathbf{r}';0)\,d\mathbf{r}'\Big\}A_\nu(\mathbf{r}) \equiv -\frac{\rho_s}{\rho}\,\frac{ne^2}{m}\delta_{\mu\nu}A_\nu(\mathbf{r}) \tag{34}$$

and we are back to the London case (though in general with a different superfluid function ρ_s/ρ than 1, even at $T=0$). Then the true $T=0$ penetration depth $\lambda(0)$ is given by

$$= \left[\frac{\rho}{\rho_s(0)}\,\frac{m\epsilon_0}{ne^2}\right]^{1/2} \equiv \left[\frac{\rho}{\rho_s(0)}\right]^{1/2}\lambda_{\rm L}(0), \quad \lambda_{\rm L}(0) \equiv \sqrt{\frac{m\epsilon_0}{ne^2}} \tag{35}$$

which is the London value $\lambda_{\rm L}(0)$ for the clean case and $\lambda_{\rm L}(0) \times (\xi_0/l)^{1/2}$ for the dirty case. More generally, alloying increases the penetration depth; in fact, almost all superconducting alloys are type-II.

If $\xi_{\rm eff} \equiv (1/l + 1/\xi_0)^{-1} \gtrsim \lambda$, we have to do a self-consistent calculation (cf. lecture 3 and problem 1.2 part (d)). In the extreme anomalous limit $\xi_{\rm eff} \gg \lambda$ we can get an order of magnitude estimate by arguing that the integral of K over R is effectively cut off at a length $\sim \lambda$. Thus the factor l/ξ_0 is replaced by λ/ξ_0. Hence (since λ^{-2} is proportional to this integral)

$$\lambda^{-2}(0) \sim \lambda_{\rm L}^{-2}(0)[\lambda(0)/\xi_0] \quad \Rightarrow \quad \boxed{\lambda(0) \sim [\lambda_{\rm L}^2(0)\xi_0]^{1/3}} \tag{36}$$

Thus in the Pippard limit, $\min(\xi_0, l) \gg \lambda_{\rm L}$, we find that λ is increased by a factor $\sim \left(\xi_0/\lambda_{\rm L}\right)^{1/3}$, irrespective of whether the sample is clean or dirty.

Finite-temperature generalization (state only)

(1) The *pair radius* ξ_p is only weakly temperature-dependent, and remains $\sim \xi_0$ for a clean superconductor and $\sim (\xi_0 l)^{1/2}$ for a dirty one.

(2) The superfluid density is proportional to $(1 - T/T_c)$ for $T \to T_c$ with the constant of proportionality $\sim \rho_s(0)$, i.e. ρ for the clean limit and $\sim (l/\xi_0)\rho$ for the dirty one.

(3) The range $\xi_{\rm eff}$ of the EM kernel (current-vector potential relation) $K(\mathbf{r}, \mathbf{r}'; 0)$ remains (like the pair radius) of the same order as at $T = 0$, i.e. $\sim \xi_0$ for the clean limit and $\sim l$ for the dirty one.

Since $\lambda_{\rm L}(T) \propto \rho_s(T)^{-1/2}$, results (2) and (3) together imply that in the limit $T \to T_c$ we *always* have $\xi_0(T) \sim \xi_0 \ll \lambda_{\rm L}(T)$, i.e. we are always in the London limit. (Do not confuse this statement with the observation that the Ginzburg-Landau healing (coherence, correlation) length $\xi(T)$, which is $\sim \xi_p(1 - T/T_c)^{-1/2}$, may be either large or small compared to $\lambda_L(T)$, thereby defining type-I or type-II behavior: cf. lecture 3)