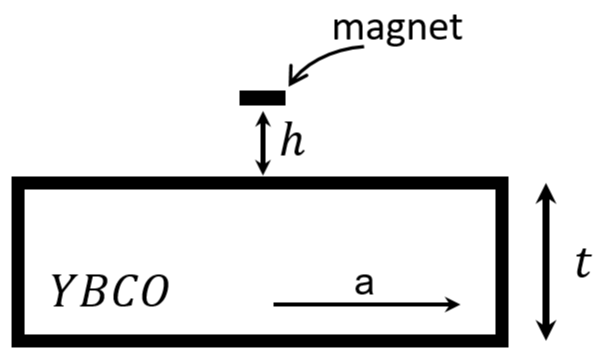
Problem Sheet 1

Note: No credit will be given for strings of algebra unaccompanied by adequate explanation, even if the final result is correct.

1. Superconducting levitation.

Note: To obtain the exact magnetic behavior in these levitation experiments would require a quite complicated calculation. However, in this problem you are mainly asked for **order-of-magnitude estimates**, and to obtain these it is legitimate (though not obviously so!) to assume that the order of magnitude of the magnetic field in the vicinity of the pellet when it is partly or wholly in the superconducting phase is not grossly different from what it would be were it (the pellet) in the normal phase.



1. Consider the demonstration of magnetic levitation which you witnessed in lecture 2. Let's take the radius a of the YBCO pellet to be 1 cm., its thickness t to be 5 mm, the mass of the permalloy magnet to be 50 mg and the height h at which it floats at the boiling temperature of liquid nitrogen (77K) to be 2 mm. (these are my best "eyeball" guesses).

(a) From these data,find a rigorous lower limit on the condensation energy of YBCO at 77K.* Assuming that the behavior of the specific heat of YBCO as a function of temperature is similar to that of the classic superconductor, estimate a lower limit on the condensation energy at $T = 0$.

(b) Using the known properties of permalloy (you may need to look these up) make a rough estimate of the maximum magnetic field which the

*You may assume that both YBCO in the normal phase and the substrate are non-magnetic. Hint: any distortion of the magnetic field lines away from their free-space form will cost a positive energy.

suspended magnet would exert on any part of the pellet were the latter entirely in the normal phase.

(c) Given that the (c-axis) lower critical field of YBCO at 77K is about 20 mT and its upper critical field about 30 T (ballpark estimates), is the bulk of the pellet likely to be in the Meissner state,the mixed state or the normal state?

(d) Make a very rough estimate of the number of vortices penetrating the pellet, and of their "typical" mutual spacing.

(e) Now consider the sumo wrestler Tosanoumi. While it isn't very clear from the picture, let's assume he is standing on a permalloy slab of area $0.1m^2$, suspended above an YBCO plate of equal area and above 1 cm thickness; we are told that the combined weight of Tosanoumi and the permalloy slab is approximately 200 kg.

Repeat the calculation of part (a) so as to obtain a second rigorous lower limit on the condensation energy of YBCO at 77 K. Look up the actual value of the condensate energy density of YBCO as directly measured and compare with the values you have calculated.

(f))(optional,for bonus points): Can you suggest (a) reason(s) why the numbers calculated in parts (a) and (e) are so different (if they are) from one another and from the experimental value?

2. Nonlocal electrodynamics of normal metals.

Consider the dynamics of the conduction electrons in a metal in a situation where the (local) electric field $\boldsymbol{\mathcal{E}}(\mathbf{r}, t)$ varies in both space and time. Let $\delta n(\mathbf{p}, \mathbf{r} : t)$ be the deviation of the semiclassical distribution function $n(\mathbf{p}, \mathbf{r} : t)$ from its thermal equilibrium form $f_0(\varepsilon_{\mathbf{p}})$ where $f_0(\varepsilon)$ is the Fermi function.† The linearized Boltzmann kinetic equation may be taken for our purposes to be of the form

$$\frac{\partial}{\partial t}\,\delta n(\mathbf{r}, \mathbf{p} : t) = -\mathbf{v}_{\mathbf{p}} \cdot \boldsymbol{\nabla} \delta n(\mathbf{r}, \mathbf{p} : t) - e\left[\frac{\partial f_0}{\partial \varepsilon_{\mathbf{p}}}\right] \mathbf{v}_{\mathbf{p}} \cdot \boldsymbol{\mathcal{E}}(\mathbf{r}, t) - \frac{\delta n(\mathbf{r}, \mathbf{p} : t)}{\tau} \quad (1)$$

where $\mathbf{v}_{\mathbf{p}} \equiv \mathbf{p}/m$ and τ is a phenomenological collision time.

†Assume that $k_{\mathrm{B}}T \ll \varepsilon_{\mathrm{F}}$ so that the usual expansion around the Fermi energy is justified.

(a) Show that the solution of this equation, up to additive transients, is of the form

$$\delta n(\mathbf{r}, \mathbf{p} : t) = -e \frac{\partial f_0}{\partial \varepsilon_{\mathbf{p}}} \int_{-\infty}^{t} dt' \, \mathbf{v} \cdot \boldsymbol{\mathcal{E}}\big(\mathbf{r} - \mathbf{v_p}(t-t'), t'\big) \exp -(t-t')/\tau \quad (2)$$

and interpret this result physically. Hence, obtain an expression for the electric current $\mathbf{j}(\mathbf{r}, t)$ in terms of $\boldsymbol{\mathcal{E}}(\mathbf{r}', t')$.

(b) Consider the case of a sinusoidally varying local field,

$$\boldsymbol{\mathcal{E}}(\mathbf{r}, t) = \boldsymbol{\mathcal{E}}_0(\mathbf{r}) \exp -i\omega t \quad (3)$$

By introducing the variable $\mathbf{r}'(\mathbf{p}, t-t') \equiv \mathbf{r} - \mathbf{v_p}(t-t')$, show that the Fourier transform $\mathbf{j}(\mathbf{r}, \omega)$ of the current can be written in the Chambers form[‡]

$$\mathbf{j}(\mathbf{r}, \omega) = e^2 \left[\frac{dn}{d\varepsilon}\right] \frac{v_{\mathrm{F}}}{4\pi} \int d\mathbf{r}' \, \frac{\mathbf{R}\big(\mathbf{R} \cdot \boldsymbol{\mathcal{E}}_0(\mathbf{r}')\big)}{R^4} \exp -i\omega R/v_{\mathrm{F}} \; \exp -R/l \quad (4)$$

where $\mathbf{R}$ is a shorthand for $\mathbf{r} - \mathbf{r}'$, v_{F} is the Fermi velocity, $dn/d\varepsilon$ the density of states (of both spins) at the Fermi surface and $l \equiv v_{\mathrm{F}}\tau$ the mean free path. "Note: you may need to use various prescriptions, e.g. for converting sums over p into integrals over energy and angle, etc. You should have met these in your basic solid state course, but if not, look them up in some standard text.

(c) Show that in the limit where $\boldsymbol{\mathcal{E}}_0(\mathbf{r})$ is slowly varying over distances of the order of both ℓ and $v_{\mathrm{F}}/\omega \equiv \lambda_\omega$, the Chambers formula reduces to the 'local' form

$$\mathbf{j}(\mathbf{r}, \omega) = \sigma(\omega) \boldsymbol{\mathcal{E}}(\mathbf{r}, \omega) \quad (5)$$

where the conductivity $\sigma(\omega)$ is given by the Drude expression:

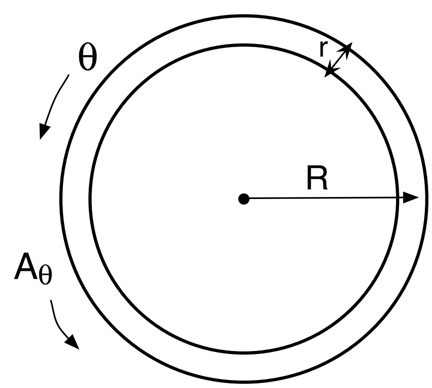
$$\sigma(\omega) = \frac{1}{3} e^2 v_{\mathrm{F}} l \, \frac{dn}{d\varepsilon} \frac{1}{1 + i\omega\tau} = \frac{ne^2\tau}{m} \frac{1}{1 + i\omega\tau} \quad (6)$$

where the last expression is valid for a free-electron gas (Sommerfeld model). Use this result to rewrite the prefactor in the Chambers formula in terms of the plasma frequency (0).

[‡]You may assume without proof that the correct prescription for the transformation from the integrals over t' and the direction of $\mathbf{p}$ to that over $\mathbf{r} - \mathbf{r}'$ is given by $\int dt' \int d\Omega_{\mathbf{p}} \to \int \frac{d(\mathbf{r}-\mathbf{r}')}{v_{\mathrm{F}} |\mathbf{r}-\mathbf{r}'|^2}$

(d) By combining Chambers' equation with Maxwell's equations, show that the problem of penetration of a (transverse) EM field into the metal is determined, in the limit where the free-space wave length $2\pi c/\omega$ is long compared to everything else in the problem, by three characteristic lengths, namely the quantities l and λ_ω and the 'high-frequency skin depth' $\delta_0 \equiv (m/ne^2\mu_0)^{1/2}$ ($\equiv \lambda_{\mathrm{L}}(0)$ if the system becomes superconducting). By a self-consistent dimensional argument, or otherwise, find the dependence (apart from numerical factions of the actual penetration depth $\delta(\omega)$ on l, λ_ω and δ_0 in the limits (i) $\lambda_\omega \gg \delta_0 \gg l$ and (ii) $l \gg \lambda_\omega \gg \delta_0$. Hint: Assume that the field falls off as exp $\delta(\omega) * z$ where $\delta(\omega)$ may be complex, and that if so the real part of the exponent is at least comparable in order of magnitude to the imaginary part. It may help your intuition to look, on the way to case (ii), at the case l $\gg \delta_0 \gg \lambda_\omega$.

3. Meissner effect and flux quantization



Consider a thin metallic ring of radius R and circular cross-section πr^2. For simplicity (only) we will assume $r \ll R$ and neglect any terms of higher order than zeroth in r/R. We apply to it an external flux Φ_{ext} such that the vector potential A_{ext} is everywhere in the tangential direction and equal to $\Phi_{\mathrm{ext}}/2\pi R$ (cf. above). The effect is to replace the tangential component of momentum, p_θ, by $(p_\theta - eA_\theta)$, in both the Hamiltonian and the expression for the electric current.

(a)) By an appropriate transformation of variables, show that in classical equilibrium statistical mechanics no tangential current is induced. (Bohr-van Leeuwen theorem.)

(b) In general, a current may be induced, and will then produce an 'induced' flux Φ_{ind} which will add to the external one Φ_{ext}. Show that for $r \ll \lambda_{\mathrm{L}}(T)$ Φ_{ind} is negligible compared to Φ_{ext} provided the London equation is obeyed, and thus A may be taken constant and equal to A_{ext}.

(c) Consider the general quantum-mechanical case: write down the time-independent Schrödinger equation for the many-body system and state the boundary conditions which the wave functions must satisfy. By making an appropriate transformation on the wave function, show that the free energy must be periodic in Φ_{ext} with a periodicity $\tilde{\phi}_0 = h/e$, i.e. $F(\Phi + n\tilde{\phi}_0) = F(\Phi)$.

[Note that this result, which is quite generic, is entirely compatible with (a) $F(\Phi) = \text{const.}$ (i.e. independent of Φ) and (b) $F(\Phi) = \text{periodic}$ in some submultiple of $\tilde{\phi}_0$, e.g. $h/2e$.]

(d) Consider a system of **noninteracting** QM particles in the above geometry in the presence of external flux Φ ext. Solve the time-independent Schroedinger equation for the tangential motion and write down expressions, for each level n, for the wave function, the energy and the expectation value of the tangential current.

Hence find an expression for the total tangential current in terms of the thermal occupation factors f_n. Assuming that the replacement $\sum_n f_n \to \int f(n)\,dn$ is valid provided $|f_{n+1} - f_n| \ll f_n$, find the condition in terms of the radius R of the annulus,the mass m of the particles and the temperature, for the Bohr-van Leeuwen theorem to be approximately satisfied in this (quantum) case, assuming classical (Gibbs) statistics.[§]

Now consider the case of a Bose system below its transition temperature, so that a macroscopic fraction $f_0(T)$ of all particles must occupy the lowest energy single-particle state. Show that under these conditions the Bohr-van Leeuwen theorem is *not* satisfied and sketch the form of free energy and the current as a function of $\Phi \to \Phi_{\text{ext}}$. Show in particular that for $\Phi \to \Phi_{\text{ext}} \ll \tilde{\phi}_0/2$ the current is given by the London equation, with the superfluid fraction $n_s(T)/n$ equal to $f_0(T)$.

Solutions to be put in 598sc homework box (2nd floor Loomis) by 9 a.m. on Mon. 17 Sept.

[§] i.e. that the probability of the occupation of a given single particle state is proportional to $\exp -\beta E_n$ where E_n is the energy of the state.