# Problem Sheet 2

## 1. Pair-breaking in the Cooper problem.*

(a) Consider a system of $N-2$ particles in equilibrium at $T=0$ in a Zeeman field $\mathcal{H}$, so that the energy of a particle with momentum $\mathbf{k}$ and spin $\sigma = \pm 1$ is $\hbar^2\mathbf{k}^2/2m \mp \mu_{\mathrm{B}}\sigma\mathcal{H}$. Repeat the Cooper calculation for two 'added' particles with the 'BCS' form of interaction ($V_{\mathbf{kk'}} = -V_0$ if $\epsilon_{\mathbf{k}}, \epsilon_{\mathbf{k'}} < \epsilon_c$, 0 otherwise), and find the condition for a bound state to exist, if the spin state of the added pair is a singlet; express this condition in terms of the original ($\mathcal{H}=0$) bound state energy.

(b) If we assume instead that the spin state is a triplet (e.g. $\uparrow\uparrow$), can a bound state exist (i) for the BCS form of $V_{\mathbf{kk'}}$ (ii) for a more general form? (You are not required to find its energy.)

(c) Returning to the original ($\mathcal{H}=0$) Cooper problem, suppose that we require the added (singlet) pair to have finite com momentum $\hbar\mathbf{K}$. What is the maximum value of $\mathbf{K}$ for which a bound state exists? (assume $|\mathbf{K}| \ll \epsilon_c/\hbar v_F$.)

(d) Consider a metal containing a nonzero concentration of (nonmagnetic) impurities ('alloyed'). The single-particle eigenstates are still eigenstates of $\sigma$; they are no longer eigenstates of $\mathbf{k}$, but any state $|n,\uparrow\rangle$ will still have a 'time-reversed' partner $|\bar{n},\downarrow\rangle$ which is degenerate with it ($\epsilon_{\bar{n}\downarrow} = \epsilon_{n\uparrow}$). Thus, the natural ansatz is to pair $|n,\uparrow\rangle$ with $|\bar{n},\downarrow\rangle$. Assuming that the matrix element for scattering $(n\uparrow, \bar{n}\downarrow) \to (n'\uparrow, \bar{n}'\downarrow)$ still has the BCS form (i.e. constant for $|\epsilon_n|, |\epsilon_{n'}| < \epsilon_c$, zero otherwise), repeat the Cooper calculation and find the bound state energy in terms of $V_0$, $\epsilon_c$ and single-particle DoS $N(0) \equiv \sum_n \delta(\epsilon - \epsilon_n)$. If we assume that the last quantity is not appreciably affected by alloying, what inference might we reasonably draw about the effect of nonmagnetic impurities on (BCS) superconductivity?

*Assume throughout this problem that the cut off energy $\epsilon_c$ is much larger than both the Zeeman splitting and the $\mathcal{H}=0$ bound-state energy. You may assume the result that the average over the unit sphere of $|\ell n||\cos\theta||$ is 1.

(e) † (Optional, for bonus points): Suppose that $\mathcal{H}$ is a little above the threshold field calculated in part (a). Is it possible, nevertheless, to form a bound pair by giving it finite linear COM momentum $\mathbf{K}$? If so, what is (approximately) the best choice of $|\mathbf{K}|$? Does the direction matter? What is the spin state of the pair?

## 2. Off-diagonal long-range order.

Consider the quantity

$$K_{\alpha\beta\gamma\delta}(\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3\mathbf{r}_4) \equiv \langle \psi^\dagger_\alpha(\mathbf{r}_1)\psi^\dagger_\beta(\mathbf{r}_2)\psi_\gamma(\mathbf{r}_3)\psi_\delta(\mathbf{r}_4)\rangle$$

(a) Find an expression for $K$ for a noninteracting Fermi gas in thermal equilibrium at a temperature $\ll T_{\mathrm{F}}$, and in particular give an argument* to suggest that it vanishes in the limit $|\mathbf{r}_1 - \mathbf{r}_2|$, $|\mathbf{r}_3 - \mathbf{r}_4|$ finite, $R \equiv \frac{1}{2}\,|(\mathbf{r}_1 + \mathbf{r}_2) - (\mathbf{r}_3 + \mathbf{r}_4)| \to \infty$

(b) Evaluate the expression explicitly for $T = 0, \mathbf{r}_1 = \mathbf{r}_2, \mathbf{r}_3 = \mathbf{r}_4$, and estimate how fast it vanishes as a function of $R$.

(c) Now consider a BCS superconductor at $T = 0$. Show that there is now an extra term in $K$ which is finite in the limit of part (a), for some choices of $\alpha$, $\beta$, $\gamma$, $\delta$ (which ones?).

(d) Estimate the order of magnitude of the fluctuations in the total particle number $N$ which result from the use of the BCS ground state wave function.

[Note: Part (d) is only loosely connected to the rest of the question.]

## 3. Coherence factors etc.

For some purposes, e.g. the calculation of spin diffusion, it is necessary to consider the spin current operator $\mathbf{J}^{(\alpha)}_{\mathrm{spin}}(\mathbf{r},t)$ which is defined (provided the potential is spin-independent) by the continuity equation

$$\frac{\partial S_\alpha(\mathbf{r},t)}{\partial t} + \operatorname{div}\mathbf{J}^{(\alpha)}_{\mathrm{spin}}(\mathbf{r},t) = 0$$

†In this part you may find the following result useful: The quantity $-\int \frac{d\Omega}{4\pi}\ln|1 - \alpha\cos\theta|$, regarded as a function of (positive real) $\alpha$, has a maximum at $\alpha = 1$ equal to $1 - \ln 2$.

*You may assume the result that the Fourier transform of a smooth function tends to zero for sufficiently large values of its argument where $S_\alpha(\mathbf{r},t)$ is the density of the $\alpha$-th component of spin.

(a) Write down the expression for the spatial Fourier transform of $\mathbf{J}^{(\alpha)}_{\text{spin}}(\mathbf{r},t)$ in second- quantized form (i.e., in terms of the operators $a^\dagger_{\mathbf{p}\sigma}$, $a_{\mathbf{p}\sigma}$), and show that it satisfies a sum rule similar to the $f$-sum rule (again assume spin-independence of the potential).

Consider now a BCS superconductor at $T = 0$:

(b) Can the flow of the condensate give rise to a finite contribution to $\mathbf{J}^{(\alpha)}_{\text{spin}}$? Why (not)?

(c) * Find an expression for the (Fourier-transformed) response function of $\mathbf{J}^{(\alpha)}_{\text{spin}}$ in terms of the energy gap and the normal-state energies. (Hint: use the Bogoliubov transformation), and evaluate it in the limit $\omega = \mathbf{k} = 0$.

(d) Discuss qualitatively the behavior of the 'longitudinal' and 'transverse' spin current correlation function in the $T \to 0$, static, long-wavelength limit, and compare with that of the (electric) current correlation function. What is the fundamental reason for the differences?

[In parts (c-d), you are recommended to choose your spin axes so that $\alpha$ corresponds to $z$.]

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Solutions to be put in 598SC homework box (2nd floor Loomis)
by 1 p.m. on Mon. 1 Oct.

*Before attempting this part of the question you may find it helpful to read the discussion of coherence factors in lecture 8 (or Tinkham section 3.9)