Problem Sheet 4

1. 'Meissner' and 'Pauli' upper critical fields.

Consider a BCS superconductor with no magnetic impurities or spin-orbit coupling, at a temperature T near T_c in an external field $\mathbf{H}$.

(a) *'Meissner' upper critical field.*

For this part of the problem, ignore the coupling of the electron spins to the field and use the GL formalism, ignoring the fourth-order term on the grounds that it will not affect the existence or not of a solution. Show that the condition for a nonzero order parameter to be thermodynamically stable is

$$H < \frac{\Phi_0}{2\pi\xi^2(T)} \equiv H_{c2}^{\text{Meissner}} \tag{1}$$

where $\xi(T)$ is the GL healing length.

[Hint: Either use known results on the QHE, etc. (but watch factors of 2!) or use the 'radial gauge' $\mathbf{A}(\mathbf{r}) = \frac{1}{2}(\mathbf{r} \times \mathbf{B})$ and the fact that the lowest eigenvalue λ_n of the equation

$$-\frac{1}{r}\frac{d}{dr}r\frac{df}{dr} + \frac{1}{r^2}(n - r^2/2)^2 f = \lambda_n f$$
$$(n = 0, 1, 2, 3 \ldots)$$

is 1 independently of n.]

(b) *'Pauli' critical field.*

For this part, ignore the orbital coupling treated in part (a) and consider only the 'Zeeman' coupling to the spins. An order-of-magnitude estimate of the largest field which the superconducting state can tolerate in the presence of this interaction is obtained by equating the superconducting condensation energy in zero field to the loss of polarization energy due to formation of (singlet) pairs; to estimate the latter it is adequate to neglect the nonlinearity of the susceptibility. Show that in this way we obtain for $T \to T_c$ the result

$$H < \frac{k_{\rm B} T_c}{\mu_{\rm B}} \left[A'(1 - T/T_c)\right]^{1/2} \equiv H_{c2}^{\rm Pauli} \tag{2}$$

and calculate the constant A'. Compare the latter with the 'true' value $A = 4\pi^2/7\zeta(3)$ obtained from a calculation of the instability of the normal phase, and comment briefly on a possible reason for any discrepancy.

[Note: $1 - Y(T) \to 2(1 - T/T_c)$ for $T \to T_c$ where $Y(T)$ is the Yosida function.]

(c) Assume now that for the purpose of order-of-magnitude estimates the formulae obtained for H_c in parts (a) and (b) can be extrapolated to arbitrary values of T/T_c. Are 'Pauli' effects ever important in the limit $T \to T_c$? Are they likely to be important for any T for

(i) a clean BCS superconductor
(ii) a very dirty BCS superconductor
(iii) heavy-fermion systems ($T_c \sim 1$K, $H_{c2} \sim 1 - 10$T)
(iv) cuprates with $\mathbf{B} \parallel \hat{\mathbf{c}}$ ($T_c \sim 100$K, c-axis $H_{c2} \sim 100$T)

(d) Generalize the result of part (a) to the case where the coefficients γ, and hence the healing lengths, are different for the two directions perpendicular to the field. (see e.g. Tinkham Ch.9 or AJL section 7.10) Assuming that the Pauli effect remains isotropic, estimate whether it is likely to be important for fields on the cuprates parallel to the ab-plane (estimated H_{c2} for this orientation $\sim 10^3$T.)

(Note in parts (c)(iii-iv) and (d) you are not required to derive the quoted values of H_c2 from part (a), but may use them directly)

2. 'Toy model' to illustrate some aspects of the BdG equations.

Consider the Hamiltonian

$$\hat{H} = (\lambda\, a_1^\dagger a_2 - i\mu\, a_1^\dagger a_2^\dagger) + \text{h.c.} \tag{3}$$

where a_i's are fermion operators with the standard anticommutation relations, and the parameters λ, μ are real and non zero. Evidently the relevant Hilbert space is 4D and spanned by the vectors $|n_1, n_2\rangle$, $n_1, n_2 = 0, 1$. Note that $\hat{H}$, while not conserving the quantity $\hat{n}_1 + \hat{n}_2$, does conserve its parity (i.e. its "evenness" or "oddness").

(a) By considering the quantity $\hat{H}^2$, or otherwise, find the eigenvalues of $\hat{H}$. Is there ever any degeneracy?

(b) Find the even-parity eigenstates of $\hat{H}$ explicitly as linear combinations of $|00\rangle$ and $|11\rangle$, and express them in the form $(\alpha + \beta\, a_1^\dagger a_2^\dagger)|00\rangle$. Similarly for the odd-parity states. If $|\mu| > |\lambda|$, which of the four states is the groundstate?

(c) Find two linear combinations of the a's and $a^\dagger$'s which annihilate the even-parity ground state, and two which create from it (normalized) odd-parity states.

(d) Now write the Bogoliubov quasiparticle creation operator $\gamma_n^\dagger$ in the form $\gamma_n^\dagger = \sum_{i=1,2}(u_i^{(n)} a_i^\dagger + v_i^{(n)} a_i)$. By demanding that $[\hat{H}, \gamma_n^\dagger] = E_n \gamma_n^\dagger$, or otherwise, derive the 'BdG' equation. Solve for the u_i's and v_i's for each n and compare with the results of part (c). (the algebra to solve for the $u's$ and $v's$ is simplified by noting (a) that $u_1^{(n)}$ can be taken $= 1$ without less of generality (b) by inspection, $u_1^{(n)}$has to be ± 1 and $v_{1,2}^{(n)} \pm i$)

(e) In the special case $\lambda = \mu$, show that one of the odd-parity states is degenerate with the even-parity groundstate. By switching to the basis $|+\rangle \equiv 2^{-1/2}(|1\rangle + |2\rangle)$, $|-\rangle \equiv 2^{-1/2}(|1\rangle - |2\rangle)$, or otherwise, interpret this result physically.

3. Anomalous ('π') Josephson junction

Consider a tunnel-oxide junction containing magnetic impurities: for simplicity assume them to be polarized at random in the $\pm z$ directions. Then the transmission matrix element $T_{\mathbf{kq}\sigma}$ may in general depend on σ: let us write

$$T_{\mathbf{kq}\sigma} = A_{\mathbf{kq}} + \sigma B_{\mathbf{kq}} \tag{4}$$

where A and B are assumed to satisfy $A_{\mathbf{kq}} = A^*_{-\mathbf{k}-\mathbf{q}}$, $B_{\mathbf{kq}} = B^*_{-\mathbf{k}-\mathbf{q}}$ and the quantity $\overline{A^*B}$ is zero. Assume that the two bulk superconductors connected by the junction are of simple BCS type with s-wave pairing and that $T = 0$.

(a) Rederive the expression for the Josephson coupling in the form

$$E_{\mathrm{J}} = -\frac{I_c \Phi_0}{2\pi} \cos \Delta\phi \tag{5}$$

and show that under suitable circumstances (what are they?) the quantity I_c can be *negative*. What relation, if any, can you now obtain between I_c and the normal-state junction resistance?

(b) Consider an "rf SQUID" device in which a single such junction, with a negative value of I_c, is inserted in a bulk superconducting ring (thickness $\gg \lambda$). Show that the total energy of the ring is, including self-inductance effects, has the following form as a function of the total trapped flux Φ (which in general includes the contribution $LI, L =$ self-inductance):

$$E(\Phi) = +\frac{|I_c|\Phi_o}{2\pi}\, cos 2\pi\Phi/\Phi_o \;+\; (\Phi - \Phi_{ext})^2/2L \quad \Phi_{ext} = \text{externally applied flux.} \tag{6}$$

Hence show that for $L \to \infty$ there are two degenerate groundstates related by time reversal. What is the energy barrier between them?

(c) Now consider the effect of the finite self-inductance L of the ring (but assume zero external flux). Show that below a threshold value L_c, of L which depends on $|I_c|$ the degeneracy is removed, and find L_c. Find an expression for the height of the barrier for L just above L_c.

(d) Consider specifically a ring with self-inductance 0.1 nH, junction critical current $|I_c|=$ 4.4 μA and junction capacitance 25 pF. Make a rough estimate of the rate of barrier crossing by thermal activation at (i) 100 mK, (b) 10 mK. Using the result that for a quartic barrier and no 'detuning' by external noise, etc., the oscillation rate by quantum tunneling is of order $\omega_0 \exp -(16V_0/3\hbar\omega_0)$ where ω_0 is the small-oscillation frequency, estimate this rate and the temperature below which it exceeds the rate of crossing by thermal activation.

[Such a device is contemplated as a possible 'qubit']

Solutions to be put in 598SC homework box (2nd floor Loomis) by 1 p.m. on Mon. 29 Oct.