Week 11: Reading Assignment, Homework Assignment

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All lecture notes, homework, demos, references, *etc*. are available on the P598AEM website. Please spend some time checking these out!

Course Organization:

A. Lectures: Tuesday & Thursday, 12:30-1:50 pm, in 136 Loomis.
B. Weekly Reading and Homework Assignments: HW due following Thursday, in class.
C. Take-Home Midterm Exam: Oct. 10th, due Oct. 17th (in lieu of P598AEM HW 7).
D. Take-Home Final Exam: Dec. 10th, due Dec. 17th.

Reading Assignment For Week 11: Please read/work through P598AEM Lect. Notes 19-21.
Homework Assignment For Week 11: See/do HW # 11 problems on following pages.

Physics 598AEM Week 11 Homework Assignment

The ***true*** length $\hat{L}$ of a rod is ***apriori unknown***. The rod made of a special carbon fiber hybrid composite material that has ***zero*** coefficient of thermal expansion. The length of this carbon-fiber rod is ***independently measured*** using two dial-caliper rulers that are made of ***different*** materials, such that their ***thermal expansion coefficients*** $c_1 > c_2$ are different. Both dial-caliper rulers have been calibrated to give accurate results at a temperature T_0, but at any other temperature, a ***corrected estimate*** $L(T)$ of the ***true*** (but ***apriori unknown***) length $\hat{L}$ of the carbon graphite rod ***must*** be obtained using $L_i(T) = L_i^{meas} + c_i(T - T_0)$ where the index i = 1, 2 refers to dial-caliper ruler 1, 2 (respectively); L_i^{meas} is the (temperature ***uncorrected***) measured "apparent" length of the rod, *i.e*. measured using dial-caliper ruler # i = 1, 2 and $c_1 > c_2 > 0$ are the ***apriori known/specified coefficients of thermal expansion*** of the respective dial-caliper ruler materials (SI units: $m/^oC$).

The two dial-caliper rulers have (***apriori known***) Gaussian/normally-distributed 1-standard deviation length measurement ***uncertainties*** $\sigma_{L_1} \neq \sigma_{L_2} \neq fcn(T)$.

The experimentally-measured ***temperature*** T^{meas} (*n.b.* ***here***, not $= T_0$!) is the ***same*** for both dial-caliper measurements of the length of the carbon-fiber rod, but note that the ***temperature measurement*** has an (***apriori known***) Gaussian/normally-distributed 1-standard deviation ***uncertainty*** $\sigma_{T^{meas}}$ associated with it.

a.) Assume that $L_2^{meas} > L_1^{meas}$. Make a careful 2-D plot of the (horizontal) bands for $L_i^{meas} \pm \sigma_{L_i}$ *vs*. temperature T. Overlay on this plot the two bands of ***temperature-corrected*** $L_i(T)$ *vs*. temperature T. Do the two temperature-corrected bands have a ***positive*** or ***negative*** ***slope***, or are they (also) horizontal? Are the two temperature-corrected bands ***parallel*** to each other, or do they ***cross*** (*i.e.* ***intersect***) each other at some (*x*,*y*) point/region on this graph?

We want to obtain a ***weighted*** average $\bar{L}$ of ***temperature-corrected*** lengths $L_1(T)$ and $L_2(T)$. In order to accomplish this, we first need to determine the 2×2 ***symmetric*** ***covariance*** matrix of the ***temperature-corrected measurements*** $\underline{V}_{\underline{L}(T)}$.

b.) Show that the ***diagonal*** elements of $\underline{V}_{\underline{L}(T)}$ (*i*.e. ***variances***) are: $\sigma^2_{L_i(T)} = \sigma^2_{L_i} + c_i^2\sigma_T^2$ for i = 1, 2.

Note that the two ***temperature-corrected*** dial-caliper ruler ***measurements*** of the carbon-fiber rod $L_i(T)$ are ***unbiased***, *i.e*. the ***expectation value*** $E[L_i(T)] = \hat{L}$ for i = 1, 2.

Note that the ***expectation value*** of the ***temperature measurement*** $E[T] = \hat{T}$, the ***true*** temperature.

c.) Show that the ***expectation value*** for each of the ***measured*** lengths L_i^{meas} of the rod is: $E[L_i^{meas}] = \hat{L} - c_i\left(\hat{T} - T_0\right)$ for i = 1, 2.

d.) Show that the ***expectation value*** of the temperature² is: $E[T^2] = \hat{T}^2 + \sigma_T^2$.

e.) Show that the ***covariance matrix of the length measurements*** is: $\text{cov}\left(L_i^{meas}, L_j^{meas}\right) = \delta_{ij}\sigma^2_{L_i}$ where $\delta_{ij}\begin{cases} =1 \text{ for } i=j \\ =0 \text{ for } i\neq j\end{cases}$.

f.) Show that the ***covariance*** = ***off-diagonal*** elements of the ***temperature-corrected length measurements*** $\left(\underline{V}_{\underline{L}(T)}\right)_{12} = \left(\underline{V}_{\underline{L}(T)}\right)_{21}$ is:
$\text{cov}\left(L_1(T), L_2(T)\right) = E[L_1(T)L_2(T)] - E[L_1(T)]\cdot E[L_2(T)] = E[L_1(T)L_2(T)] - \hat{L}^2 = c_1c_2\sigma_T^2$.

g.) Determine the corresponding ***correlation coefficient*** $\rho\left(L_1(T), L_2(T)\right)$.

h.) Now use the LSQ method $\chi^2(\lambda) = \sum_{i,j=1}^{2}\left(L_i(T) - \lambda\right)\left(\underline{V}^{-1}_{\underline{L}(T)}\right)_{ij}\left(L_j(T) - \lambda\right)$ to obtain the "***best***" ***estimate*** of the length of the rod, λ^*. Note that this ***estimate*** of the length of the rod is the ***weighted average*** of the two temperature-corrected length measurements:

$$\lambda^* = \frac{\left(\sigma^2_{L_2} + \left(c_2^2 - c_1c_2\right)\sigma_T^2\right)L_1(T) + \left(\sigma^2_{L_1} + \left(c_1^2 - c_1c_2\right)\sigma_T^2\right)L_2(T)}{\sigma^2_{L_1} + \sigma^2_{L_2} + \left(c_1 - c_2\right)^2\sigma_T^2}$$

i.) Show that if $\sigma_T \to 0$ then $\rho\left(L_1(T), L_2(T)\right) \to 0$. What is λ^* in *this* limiting case?

j.) Show that if $\sigma_{L_i} \to 0$ then $\rho\left(L_1(T), L_2(T)\right) \to 1$. What is λ^* in *this* limiting case?