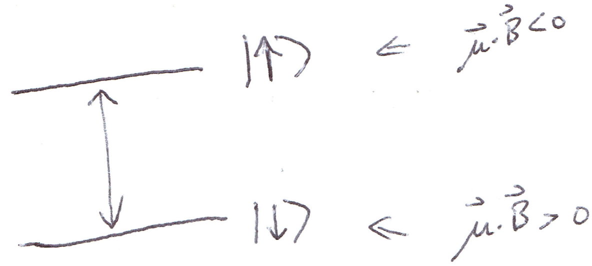
2-Level systems + classical resonance

Lecture #7
PHYS 598 A
Fall 2017

Examples of "2-level systems"

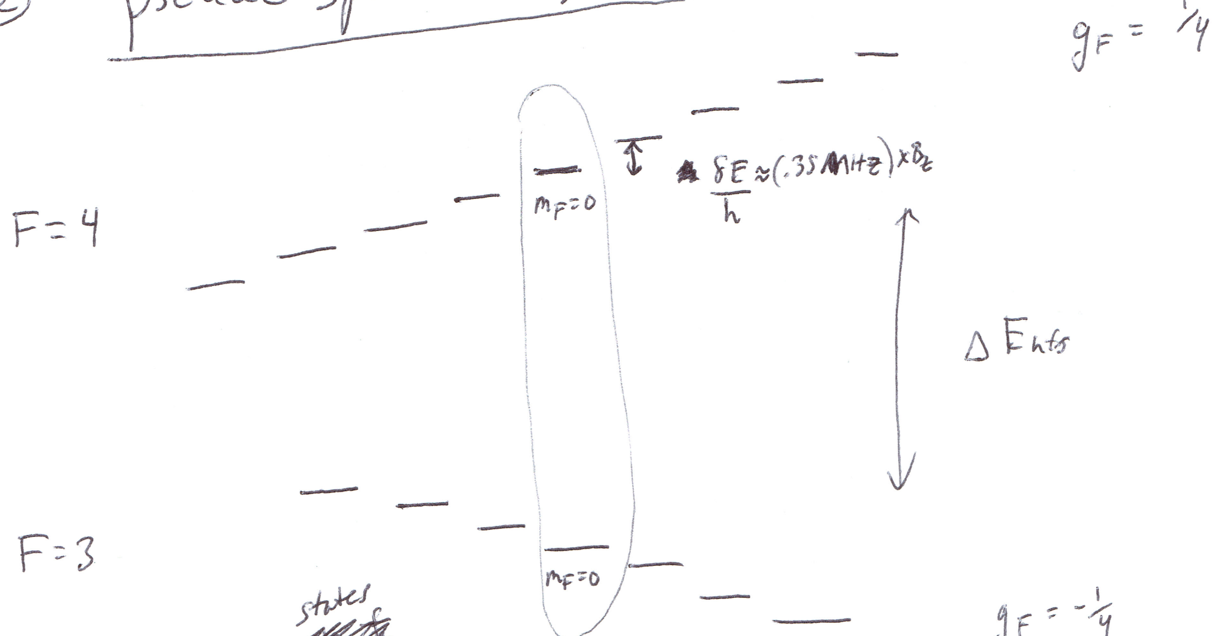
① A true spin-½ particle, i.e. electron or proton spin



$\Delta E = 2\mu B$, $\mu \to \dfrac{e\hbar}{2m}$

degenerate if $B=0$, coupled w/ transverse or oscillating B-field

② pseudo-spin-½ system (hyperfine levels)



idea: • isolate 2 ~~[illegible]~~ states out of larger Hilbert space

- off-resonant excitations have to be worried about

others

~~many, many more~~

- optical excited states (metastable)
- photon polarization states
- interferometer path
- double-well site index
- energy state of an anharmonic oscillator
- rotational / vibrational states of molecules
- many, many more

- lifetime?
- decoherence?
- special properties?
-

All of these systems can be described by the same "2-level" formalism, which we'll review now.

- spin-½ terminology
- 2-states, static coupling
- 2-states, driven

Spin-½ terminology

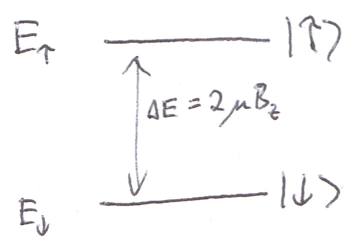
- Bloch sphere
- dressed states, etc.

spinor

general state $\Psi = \begin{pmatrix} C_\uparrow \\ C_\downarrow \end{pmatrix} = C_\uparrow |\uparrow\rangle + C_\downarrow |\downarrow\rangle$

State kets $|\uparrow\rangle, |\downarrow\rangle$

relate to excited, ground states in $B_z\hat{z}$ field (note, opposite of $\vec{\mu}$)



this will be the physical picture we have in mind, but let's start w/ general terminology

③ Define our operators of interest

$\sigma_{x,y,z}$ Pauli opr

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \frac{\hbar}{2}\hat{\sigma}_z$$

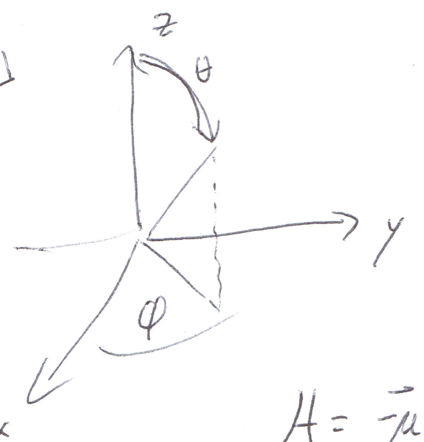
$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{\hbar}{2}\sigma_x$$

$$S_y = \frac{-i\hbar}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \frac{\hbar}{2}\sigma_y$$

we can write any state as

$$|\psi\rangle = c_\uparrow|\uparrow\rangle + c_\downarrow|\downarrow\rangle = \underbrace{\cos\frac{\theta}{2}|\uparrow\rangle + e^{i\varphi}\sin\frac{\theta}{2}|\downarrow\rangle}_{\text{polar coords}}$$

Bloch sphere



Q: What will a spin-½ particle, in the state $|\psi\rangle$, do in a uniform B-field $B_z\hat{z}$?

$$H = -\vec{\mu}\cdot\vec{B}$$

let's assume

$$\vec{\mu} = -\mu\vec{S}$$

$$\boxed{\begin{aligned} H &= \omega_L S_z \\ &= \frac{\hbar\omega_L}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)\end{aligned}}$$

$E_\uparrow = +\mu B$

$E_\downarrow = -\mu B$

$\Delta E = \hbar\omega_L = 2\mu B$

ω_L = Larmor frequency

A state like $|\psi(t=0)\rangle = c_\uparrow|\uparrow\rangle + c_\downarrow|\downarrow\rangle$

will evolve as

$$|\psi(t)\rangle = c_\uparrow e^{i\omega_L t/2}|\uparrow\rangle + c_\downarrow e^{-i\omega_L t/2}|\downarrow\rangle$$, i.e. it will

precess about the $\hat{z}$-axis

④ What if we add static transverse field? (B_x, B_y)

This is called a transverse field or coupling term, and it would relate to canting the axis of the applied field.

$$\vec{B} = B_0 \hat{z} + B_\perp \hat{x}$$

$$H = -\vec{\mu}\cdot\vec{B} = \omega_e S_z + \omega_e \frac{B_\perp}{B_0} S_x = \frac{\hbar\omega_e}{2}\begin{pmatrix} 1 & B_\perp/B_0 \\ B_\perp/B_0 & -1 \end{pmatrix}$$

where $\omega_e = 2\mu B_0/\hbar$

let's look @ limiting cases first

$\rightarrow B_\perp = 0$
$\rightarrow B_0 = 0$
$\rightarrow B_\perp \ll B_0$

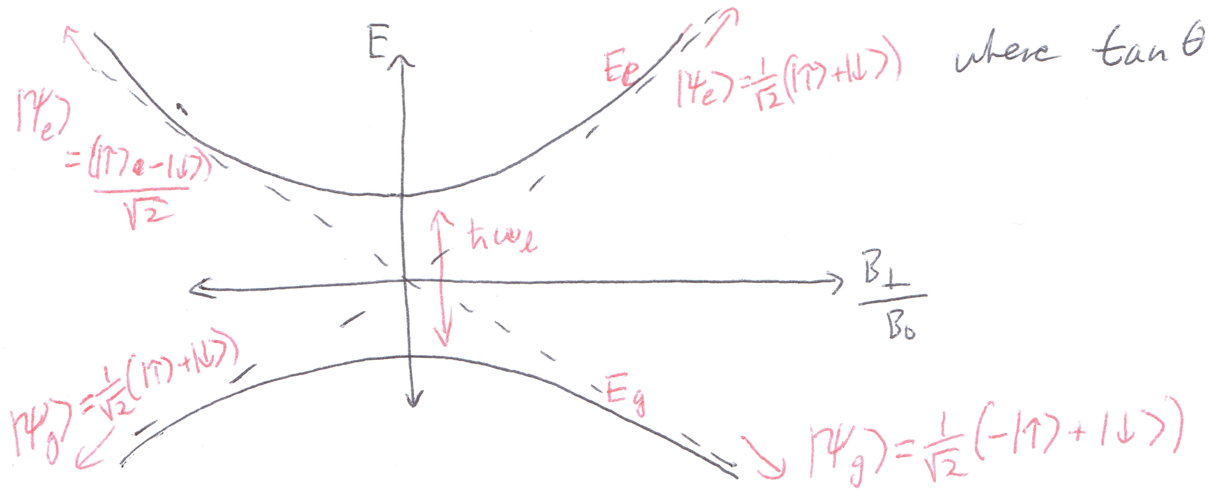
$\rightarrow$ slides

General result (Diagonalize H): $\rightarrow$ new energies, new eigenstates (dressed basis)

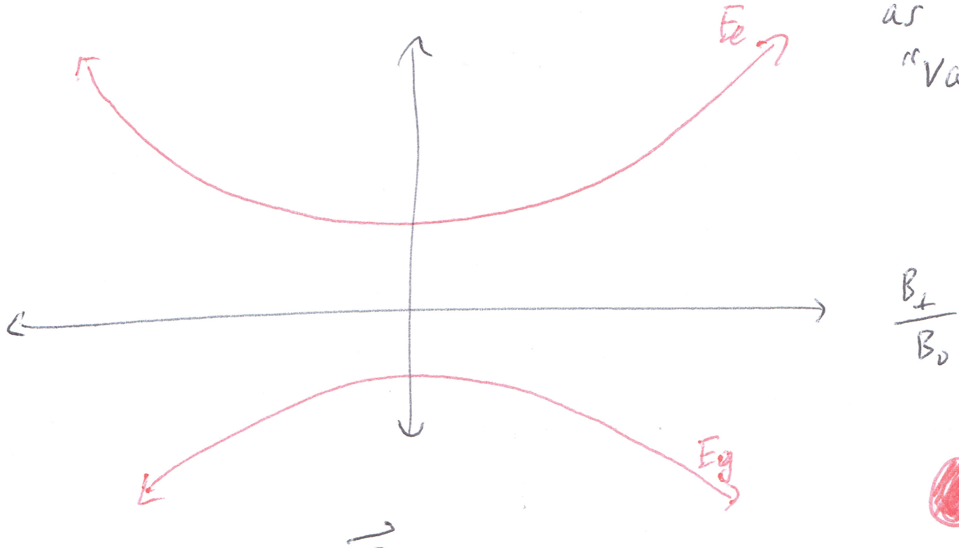
$$E_e = +\frac{\hbar\omega_e}{2}\sqrt{1 + \left(\frac{B_\perp}{B_0}\right)^2} \longleftrightarrow |\psi_e\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle$$

$$E_g = -\frac{\hbar\omega_e}{2}\sqrt{1 + \left(B_\perp/B_0\right)^2} \qquad |\psi_g\rangle = -\sin\frac{\theta}{2}|\uparrow\rangle + \cos\frac{\theta}{2}|\downarrow\rangle$$

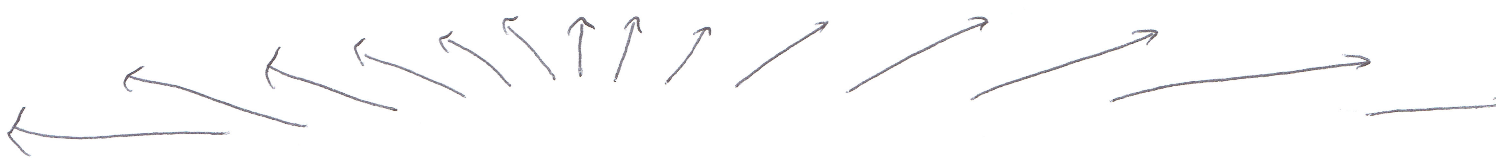
where $\tan\theta = \frac{B_\perp}{B_0}$



(5) Later in the course, we'll discuss adiabatic and non-adiabatic processes that can occur as these fields are "varied in time," relevance to magnetic trapping



$\vec{B}$



⑥ general two-level systems + classical resonance

the same formalism can be applied to generic 2-level systems, i.e, not just spin-1/2 particles.

Let's consider the eigenstates $|1\rangle$ and $|2\rangle$ of some system having energies E_1 and E_2, respectively. Let's start by ignoring any other eigenstates

$E_1 = \langle 1|H|1\rangle$
$E_2 = \langle 2|H|2\rangle$

let's add an energy offset of $\frac{E_1+E_2}{2}$ to the system to ~~...~~ allow us to use S_z's as before.

Considering added terms $\langle 1|H|2\rangle = \langle 2|H|1\rangle^* = H_{12}$ in general not being zero, we have

where $\Delta E = E_1 - E_2$

$$H = \begin{pmatrix} (E_1-E_2)/2 & H_{12} \\ H_{12}^* & -\frac{(E_1 - E_2)}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \Delta E & 2H_{12} \\ 2H_{12}^* & -\Delta E \end{pmatrix}$$

we can cast this in terms of the spin notation as

$$H = -\mu \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

where

$2\mu B_z = -(E_1 - E_2)$

i.e., $B_z = -\frac{(E_1-E_2)}{2\mu}$

$B_x = -\mathrm{Re}(H_{12})/\mu$

$B_y = \mathrm{Im}(H_{12})/\mu$

where $B_{x,y,z}$ represent effective fields

7 For these two-level systems, ~~that we~~ with states that we can write as

$|\psi\rangle = a|1\rangle + b|2\rangle$, the pseudo-spin components are given as

$$S_1 = \langle\psi|\sigma_x|\psi\rangle = a^*b + b^*a$$
$$S_2 = \langle\psi|\sigma_y|\psi\rangle = -i(a^*b - b^*a)$$
$$S_3 = \langle\psi|\sigma_z|\psi\rangle = |a|^2 - |b|^2$$

no units, normalized spin vector w/ states lying on surface of (unit) Bloch sphere

alternative description

$$|\psi\rangle = \cos\frac{\theta}{2}|1\rangle + e^{i\varphi}\sin\frac{\theta}{2}|2\rangle$$

$$S_1 = \sin\theta\cos\varphi$$
$$S_2 = \sin\theta\sin\varphi$$
$$S_3 = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos(\theta)$$

Sometimes, people use ~~S~~ ~~Sx,y,z~~

$$S_{x,y,z} = \frac{\hbar}{2} S_{1,2,3}$$

i.e. $S_z = \frac{\hbar}{2}\cos\theta$, etc.

How to deal w/ time-dependent fields? rewrite in terms of classical EoM of $\langle S_x\rangle, \langle S_y\rangle, \langle S_z\rangle$.

Note: This works well for $\langle S_z\rangle$, but otherwise fails.

Consider $|\psi\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$

- classically $\rightarrow$ $S_x = \hbar/2$, $S_z = 0$
- quantum $\rightarrow$ $\langle S_x\rangle = \hbar/2$, $\langle S_z\rangle = 0$

but $\langle \Delta S_z\rangle = \sqrt{\langle S_z^2\rangle - \langle S_z\rangle^2} = \frac{\hbar}{2}$

measurement uncertainty

(8) Classical magnetic resonance

$\vec{L}$ the angular momentum gives us the magnetic moment

$\vec{\mu} = \gamma \vec{L}$, where γ = gyromagnetic ratio

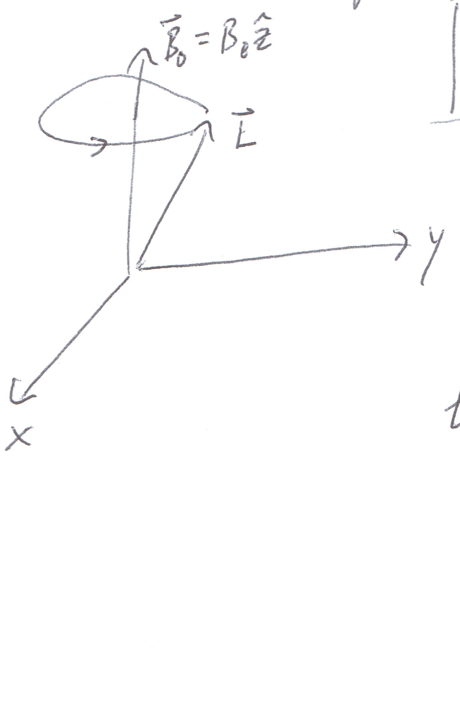
~~if we put an object w/ $|\vec{\mu}| \neq 0$ in~~

if we put such an object in a static field $\vec{B}_0$, it experiences a torque $\vec{\tau} = \vec{\mu} \times \vec{B}_0$

from classical mechanics

$$\dot{\vec{L}} = \vec{\tau} = \vec{\mu} \times \vec{B}_0 \longrightarrow \dot{\vec{L}} = \gamma \vec{L} \times \vec{B}_0$$

this is the Larmor precession from earlier



$\boxed{\text{drawn w/ } \gamma < 0}$

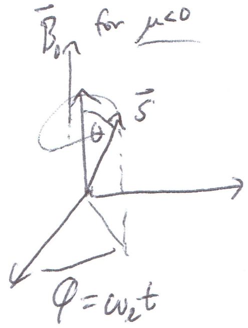
the Larmor spin-precession frequency given by

$$\omega_L = |\gamma B_0|$$

(9) in the pseudo-spin language, w/

$$|\psi(t=0)\rangle = \cos\frac{\theta}{2}|1\rangle + \sin\frac{\theta}{2}|2\rangle$$

initially along $\hat{x}$



@ later time t

$$|\psi(t)\rangle = \cos\frac{\theta}{2} e^{-i\omega_L t/2}|1\rangle + \sin\frac{\theta}{2} e^{i\omega_L \frac{t}{2}}|2\rangle$$

$$\langle S_z\rangle = \frac{\hbar}{2}\cos\theta \quad \text{(static)}$$

$$\langle S_x\rangle = \frac{\hbar}{2}\sin\theta\cos(\omega_L t)$$

$$\langle S_y\rangle = \frac{\hbar}{2}\sin\theta\sin(\omega_L t)$$

slider

~~How to go from small scale~~

What is the effect of an oscillating transverse field?

$$\vec{B}_1 = B_1\cos(\omega t)\hat{x}$$

we can solve this numerically, or go to a $\boxed{\text{rotating frame}}$ w/ rotation $\vec{\Omega}$

recall that in the rotating frame $\dot{\vec{L}}' = \dot{\vec{L}} - \vec{\Omega}\times\vec{L}'$

<u>below</u>, assume $\gamma<0$, as for quantum systems w/ $\mu<0$

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- Without the transverse field $\vec{B}_1$, one can take the rotating frame $\vec{\Omega} = \omega_\ell \hat{z}$, and $B_{eff} = 0$ as any spin vector appears stationary

- w/ the transverse field $\vec{B}_1 = B_1 \cos(\omega t)\,\hat{x}$, choose $\vec{\Omega} = \omega \hat{z}$

<u>rewrite $\vec{B}_1$</u> $\rightarrow$

$$\vec{B}_1 = \frac{B_1}{2}\left(\overbrace{\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y}}^{\text{co-rotating w/ ref. frame}}\right) + \frac{B_1}{2}\left(\underbrace{\cos(\omega t)\hat{x} - \sin(\omega t)\,\hat{y}}_{\text{Counter-rotating w/ ref. frame}}\right)$$

- let's assume

$B_1 << B_0$ (weak transverse field)

and

$|\Delta\omega| = |\omega - \omega_\ell| << \omega_\ell$ (near resonance)

In this picture, the torque vector from the <u>counter-rotating</u> term is ~~[illegible]~~ rapidly rotating around (roughly @ $2\omega_\ell$), and its time-averaged torque is essentially zero. The <u>co-rotating</u> term, on the other hand, is nearly stationary, and ~~[illegible]~~ has a major influence.

W/ this in mind, let's make a $\boxed{\text{rotating-wave approximation}}$

just ignore the counter-rotating term, as it roughly averages out

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In rotating frame $\qquad \vec{B}_1 = \frac{B_1}{2}\hat{x}'$

and $\dot{\vec{L}}' = \gamma \vec{L} \times \left(B_0 \hat{z} + \frac{B_1}{2}\hat{x}'\right) - \omega\, \hat{z} \times \vec{L}$

$= \vec{L} \times \left(\frac{\gamma B_1}{2}\hat{x}' + \omega \hat{z} + \gamma B_0 \hat{z}\right)$

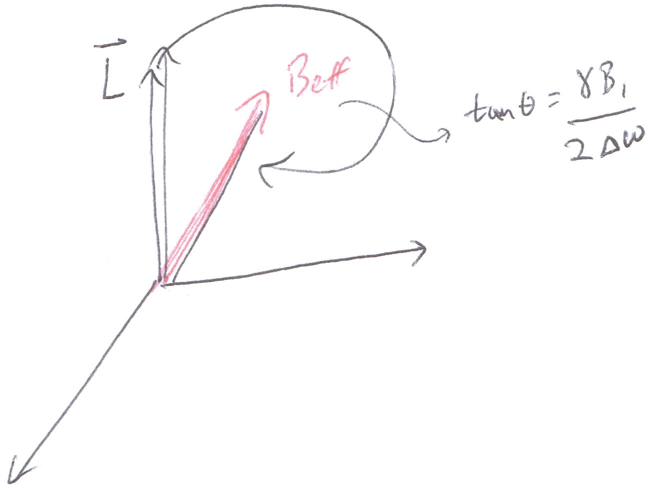
$\Delta\omega = \omega - \omega_e \Rightarrow$ detuning

$= \vec{L} \times \underbrace{\left(\frac{\gamma B_1}{2}\hat{x}' + (\omega - \omega_e)\hat{z}\right)}_{\text{effective magnetic field } \vec{B}_{eff}}$

$\omega_e = |\gamma B_0|$ and $\gamma < 0$

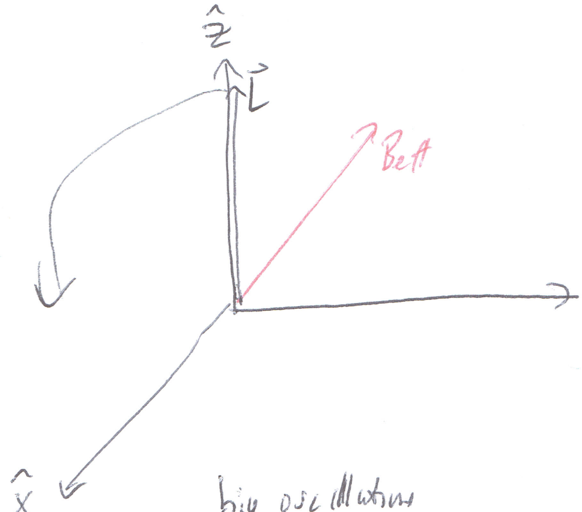
In rotating frame

if $\cancel{\;}$ $\Delta\omega = \omega - \omega_e > 0$



if $\omega = \omega_e$

$\Delta\omega = 0$



big oscillation in $L_z = \vec{L}\cdot\hat{z}$

(12)

At resonance

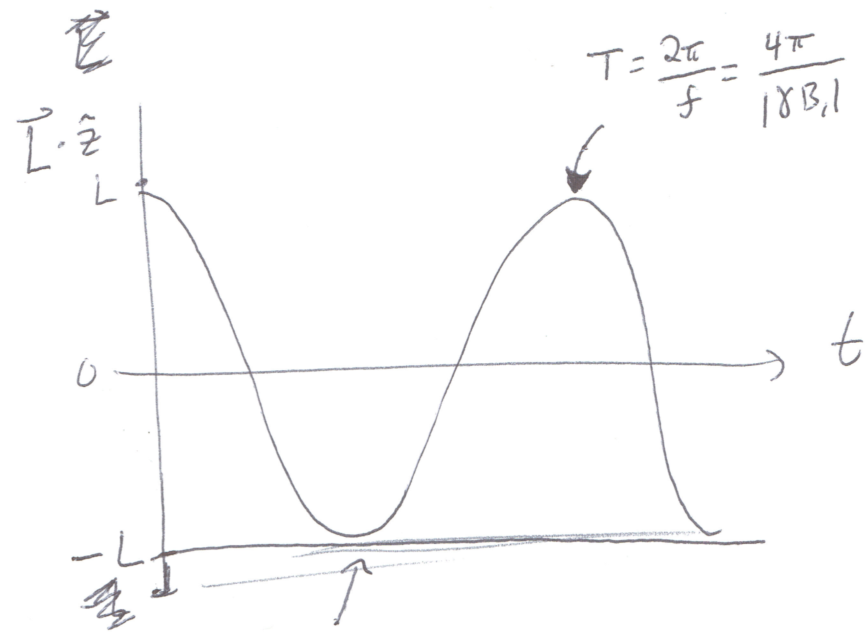
$$\vec{L}(t=0) = L\hat{z}$$

$$\dot{\vec{L}} = \vec{L} \times \frac{\gamma B_1}{2}\hat{x}'$$

$$\frac{|\dot{\vec{L}}|}{L} = \frac{|\gamma B_1|}{2}$$

$\vec{L}$ rotates in y'-z plane (about $\hat{x}'$)

Constant rotation rate → angular speed $= \frac{|\gamma B_1|}{2}$



π-pulse at $t = \frac{2\pi}{|\gamma B_1|}$

$$\vec{L} = L\hat{z} \longrightarrow -L\hat{z}$$

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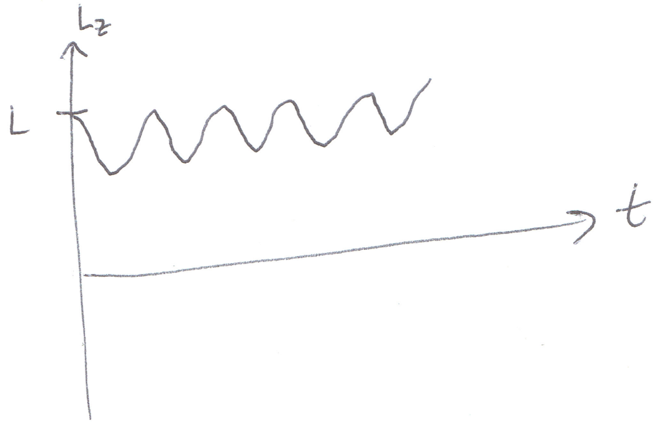
away from resonance $\Delta\omega \neq 0$

$\vec{L}$ rotates around $\vec{B}_{eff}$ @ rate $\sqrt{\left(\frac{\gamma B_1}{2}\right)^2 + (\Delta\omega)^2}$

Contrast of oscillations is reduced



Note: $\vec{B}_1 = B_1 \cos(\omega t)\hat{x}$ wasted half (the counter-rotating part) of the drive.

really we want $\vec{B}_1 = \underbrace{B_1 \cos(\omega t)\hat{x}}_{B_x} + \underbrace{B_1 \sin(\omega t)\hat{y}}_{B_y}$

$$\underbrace{H_{12} = -\mu B_1 e^{-i\omega t}}$$

More generally, the action of the driving field can be represented as

$$R = e^{-i(\vec{\sigma}\cdot\hat{n})\left(\frac{\alpha}{2}\right)}$$

where $\alpha = \sqrt{\left(\frac{\gamma B_1}{2}\right)^2 + \delta^2}\; t$ and

w/ $\delta = \Delta\omega$ the detuning

$\hat{n}$ depends on the detuning δ and the relative phase of the driving field (w.r.t. some first "pulse" / creation of a superposition)

when $\delta = 0$, $\hat{n}$ is in the $\hat{x}'-\hat{y}'$ plane,

and

with $\delta = \left|\frac{\gamma B_1}{2}\right|$, $\hat{n} = \left(\frac{\hat{z}-\hat{x}}{\sqrt{2}}\right)$

Some initial state $|\Psi_i\rangle$ is transformed by a pulse of the driving field (fixed δ, fixed B_1, duration τ) as

$$|\Psi_f\rangle = e^{-i(\vec{\sigma}\cdot\hat{n})\frac{\alpha}{2}}\,|\Psi_i\rangle \qquad \text{w/ } t = \tau$$