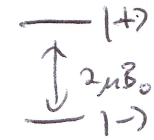
2-Level Systems cont'd

Lecture 8
PHYS 598 AQ
Fall 2017

2-Level system w/ oscillating transverse field → "quantum" version

$$H = -\vec{\mu}\cdot\vec{B} \qquad \text{let } \vec{B} = B_0\hat{z} + B_1\cos(\omega t+\varphi)\hat{x}$$



$$H = -\mu\hat{\sigma}_z B_0 - \mu B_1 \underbrace{\hat{\sigma}_x}_{\left(\frac{\hat{\sigma}_+ + \hat{\sigma}_-}{2}\right)} \underbrace{\cos(\omega t+\varphi)}_{\frac{e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}}{2}}$$

$$H = -\mu B_0\hat{\sigma}_z + \hbar\Omega(\hat{\sigma}_+ + \hat{\sigma}_-)\left[e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}\right]$$

or

$$H = \hbar\begin{pmatrix} \frac{\omega_0}{2} & \Omega\left(e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}\right) \\ \Omega\left[e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}\right] & -\omega_0/2 \end{pmatrix}$$

as in classical case, let's go to "rotating frame" and make RW

where

$$\underbrace{\hbar\omega_0}_{\text{Larmor freq}} = -2\mu B_0 \qquad \text{and} \qquad \hbar\underbrace{\Omega}_{\text{Rabi frequency}} = \frac{-\mu B_1}{4}$$

electron, $\mu<0$

Ω is real

②

"rotating frame", aka basis transformation

Let's look at time-dependence of coefficients for the state $|\psi\rangle = a_+|+\rangle + a_-|-\rangle$

Schröd

Ⓐ $$i\hbar\,\dot a_+(t) = \frac{\hbar\omega_0}{2}a_+(t) + \hbar\Omega\left[e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}\right]a_-(t)$$

Ⓑ $$i\hbar\,\dot a_-(t) = \hbar\Omega\left[e^{i(\omega t+\varphi)} + e^{-i(\omega t+\varphi)}\right]a_+(t) - \frac{\hbar\omega_0}{2}a_-(t)$$

Let's change to a new basis, w/ coefficients given by "rotating" basis

$$b_+(t) = e^{\frac{i\omega t}{2}}a_+(t) \quad \text{and} \quad b_-(t) = e^{-\frac{i\omega t}{2}}a_-(t)$$

A' $$i\frac{d}{dt}\left(e^{-\frac{i\omega t}{2}}b_+(t)\right) = \frac{\omega_0}{2}e^{-\frac{i\omega t}{2}}b_+(t) + \Omega\left(e^{-i(\omega t+\varphi)} + e^{i(\omega t+\varphi)}\right)e^{i\omega t/2}b_-$$

$$\frac{\omega}{2}e^{-\frac{i\omega t}{2}}b_+(t) + i e^{-\frac{i\omega t}{2}}\dot b_+(t) = \frac{\omega_0}{2}e^{-\frac{i\omega t}{2}}b_+(t) + \Omega e^{-\frac{i\omega t}{2}}\left(e^{-i(\omega_0 t+\varphi)} + e^{i(\omega_0 t+\varphi)}\right)e^{i\ldots}$$

counter-rotating

co-rotating

$$\hookrightarrow\ i\dot b_+(t) = \left(\frac{\omega_0-\omega}{2}\right)b_+(t) + \underbrace{\Omega\left[e^{-i(\omega_0 t+\varphi)} + \underset{\text{RWA}}{\cancel{e^{i(\omega_0 t+\varphi)}}}\right]e^{i\omega t}b_-(t)}_{\Omega e^{-i\varphi}b_-(t)}$$

Let $\delta = \omega - \omega_0$

(A) $$i\dot{b}_+ = -\frac{\delta}{2} b_+ + \Omega e^{-i\varphi} b_-$$

(B) $$i\dot{b}_- = \Omega e^{i\varphi} b_+ + \frac{\delta}{2} b_-$$

[no more time-dependence to off-diagonal coupling terms

$$\tilde{H}_{RWA} = \hbar \begin{pmatrix} -\delta/2 & \Omega e^{-i\varphi} \\ \Omega e^{i\varphi} & \delta/2 \end{pmatrix} \Rightarrow$$

recall mapping onto effective B-field

$$\mu B_x^{eff} = -Re(H_{+-})$$

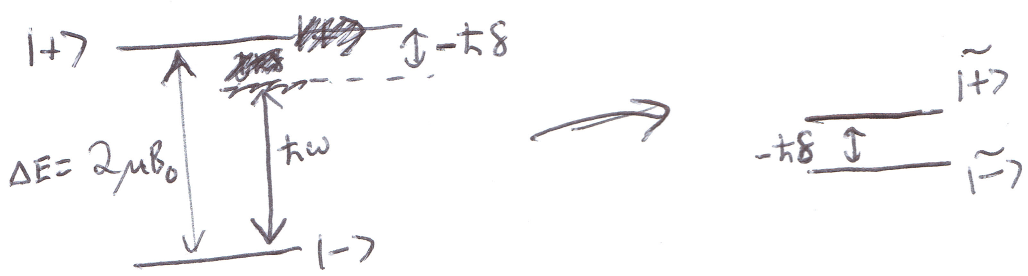
$$\mu B_y^{eff} = Im(H_{+-})$$

φ sets direction of transverse field $\Rightarrow$ rotation about

$$\hat{n}_\varphi = \underbrace{\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}}_{\text{for } \mu < 0}$$

w/ $|\tilde{\psi}\rangle = b_+(t)|\tilde{+}\rangle + b_-(t)|\tilde{-}\rangle$

where



"degenerate" if $\omega = \omega_0$

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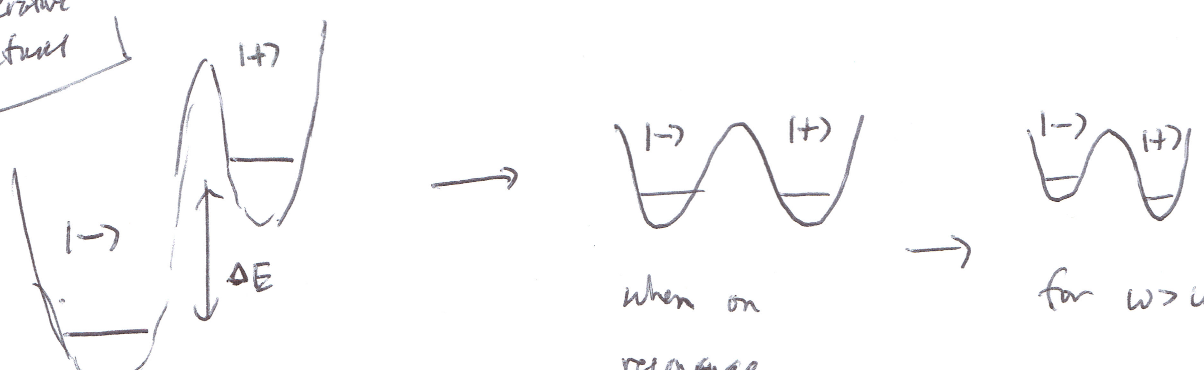
aside

alternative pictures



when on resonance

for $\omega > \omega_0$

④

general solution

eigenstates

$$|e\rangle = \cos\tfrac{\theta}{2}|\tilde{+}\rangle + \sin\tfrac{\theta}{2}e^{i\varphi}|\tilde{-}\rangle$$

$$|g\rangle = -\sin\tfrac{\theta}{2}e^{-i\varphi}|\tilde{+}\rangle + \cos\tfrac{\theta}{2}|\tilde{-}\rangle$$

w/ $\tan\theta = -\dfrac{2\Omega}{\delta}$

$$E_e = \frac{\hbar}{2}\sqrt{\delta^2+4\Omega^2} = \hbar\tilde{\Omega}$$

$$E_g = -\frac{\hbar}{2}\sqrt{\delta^2+4\Omega^2} = -\hbar\tilde{\Omega}$$

$\tilde{\Omega}$ is the generalized Rabi rate, where $\tilde{\Omega}=\Omega$ if $\delta=0$

$\Delta E = E_e - E_g = 2\hbar\tilde{\Omega}$

What happens to some initial state?

Let's say that $|\psi(t=0)\rangle = |+\rangle$ ← same as $|\tilde{+}\rangle$

$$|\psi(t=0)\rangle = \cos\tfrac{\theta}{2}|e\rangle - \sin\tfrac{\theta}{2}|g\rangle$$

not an eigenstate, and so we'll get evolution

↳ $$|\psi(t)\rangle = \cos\tfrac{\theta}{2}e^{-iE_et/\hbar}|e\rangle - \sin\tfrac{\theta}{2}e^{-iE_gt/\hbar}|g\rangle$$

w/ $$P_-(t) = |\langle -|\psi(t)\rangle|^2 = \left|\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}e^{i\varphi}e^{-i\tilde{\Omega}t} - \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}e^{i\varphi}e^{i\tilde{\Omega}t}\right|^2$$

$$= \underbrace{\left|\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}\right|^2}_{\underbrace{\left|\frac{\sin\theta}{2}\right|^2}_{\frac{\Omega^2}{4\Omega^2+\delta^2}}} \underbrace{|e^{i\varphi}|^2}_{1} \underbrace{\left|e^{-i\tilde{\Omega}t} - e^{i\tilde{\Omega}t}\right|^2}_{4\sin^2(\tilde{\Omega}t)}$$

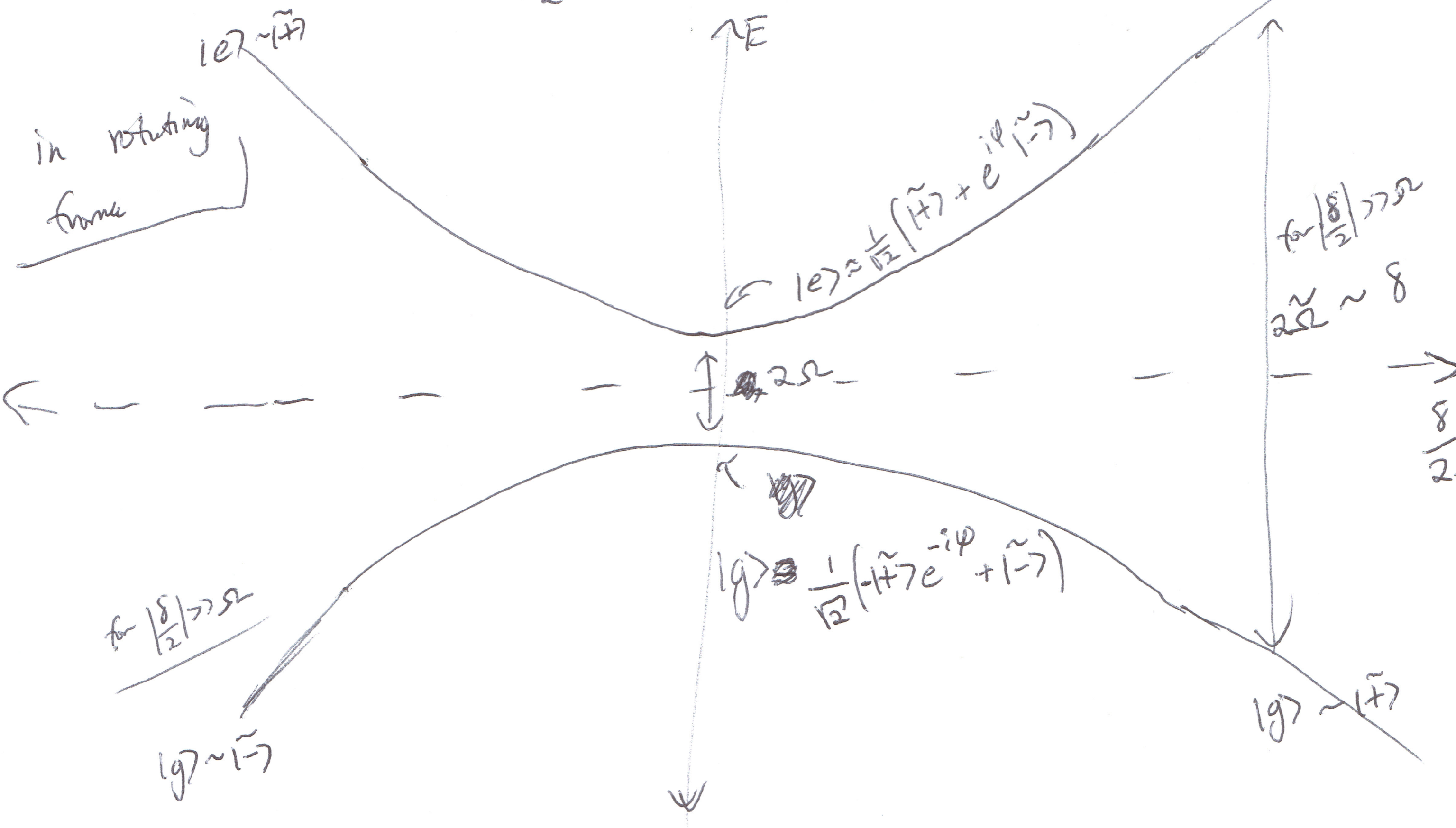
$$P_-(t) = \frac{4\Omega^2}{4\Omega^2+\delta^2}\sin^2\left[\frac{\sqrt{4\Omega^2+\delta^2}}{2}\,t\right]$$

$\boxed{\text{note}}$, no φ-dependence if starting in $|\tilde{+}\rangle$ o

or $$\boxed{P_-(t) = \left|\frac{\Omega}{\tilde{\Omega}}\right|^2 \sin^2(\tilde{\Omega} t)}$$

for starting in $|+\rangle$ @ $t = 0$.

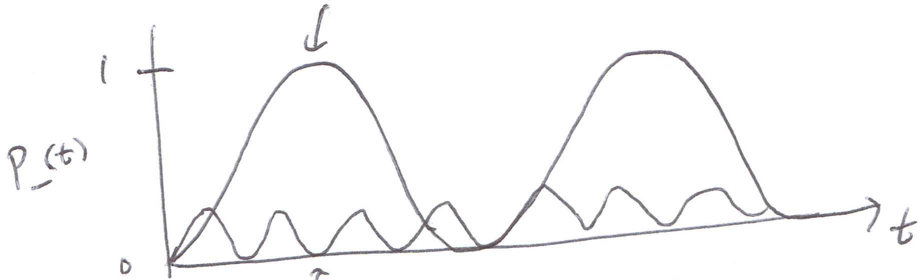
where $\tilde{\Omega} = \frac{1}{2}\sqrt{4\Omega^2+\delta^2} = \sqrt{\Omega^2 + (\delta/2)^2}$



⑥

slides on Rabi dynamics

$t_\pi = \frac{\pi}{2\Omega}$ for $\delta = 0$

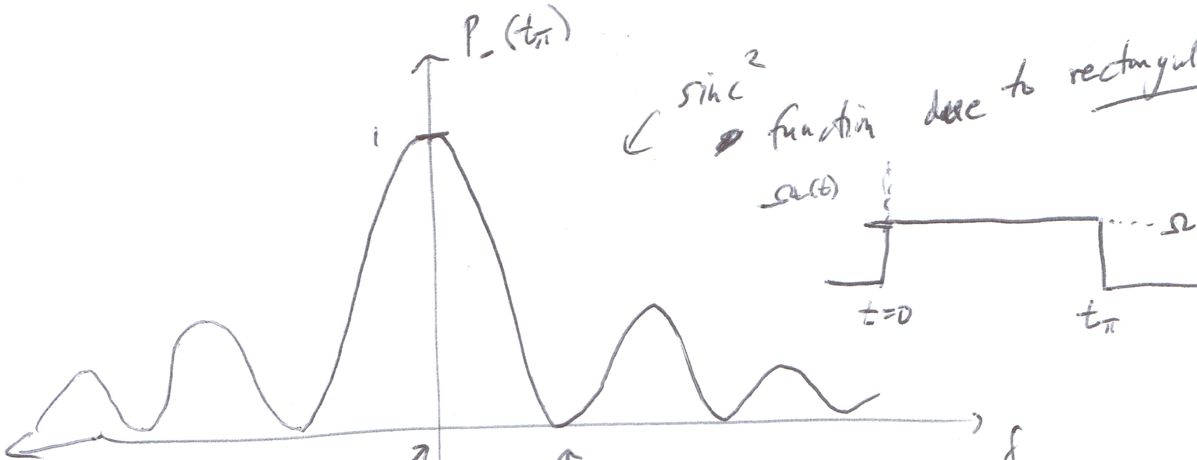


faster dynamics ($\tilde{\Omega}$),
smaller contrast $\left|\frac{\Omega}{\tilde{\Omega}}\right|^2$ for $\delta \neq 0$

some times
$\Omega \to \frac{\Omega}{2}$
in $\tilde{H}$

slides on δ-dependence : set $t = t_\pi$, scan δ

sinc^2 function due to rectangular pulse



$\tilde{\Omega} t_\pi = \frac{\pi}{2}$

$\tilde{\Omega} = \Omega$
$\delta = 0$

$$t_\pi \tilde{\Omega} = \pi$$

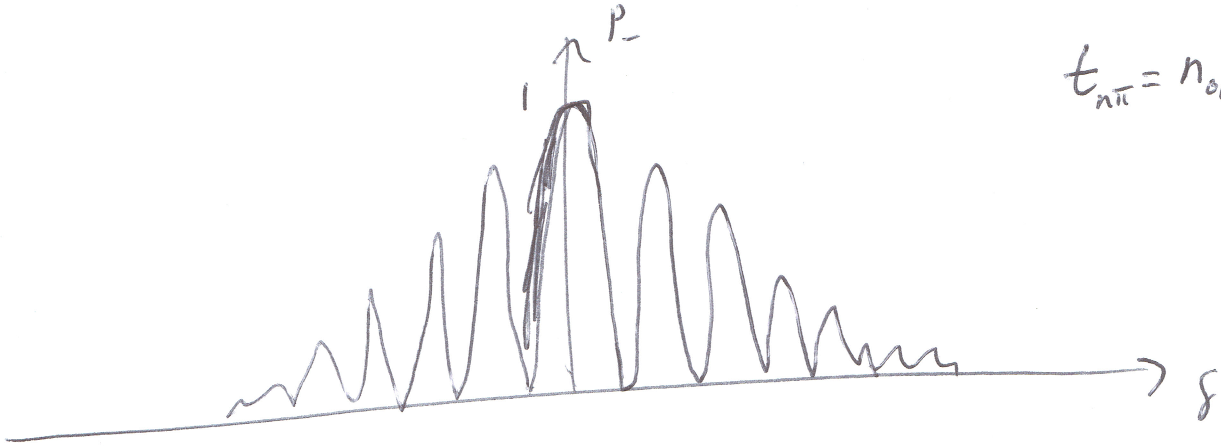
$$\frac{\pi}{2\Omega}\tilde{\Omega} = \pi$$

$$\frac{\tilde{\Omega}}{\Omega} = 2 = \frac{\sqrt{\Omega^2 + (\delta/2)^2}}{\Omega}$$

$$\delta = \pm 2\sqrt{3}\,\Omega$$

At longer times, for fixed Ω, ~~[illegible]~~ finer structure

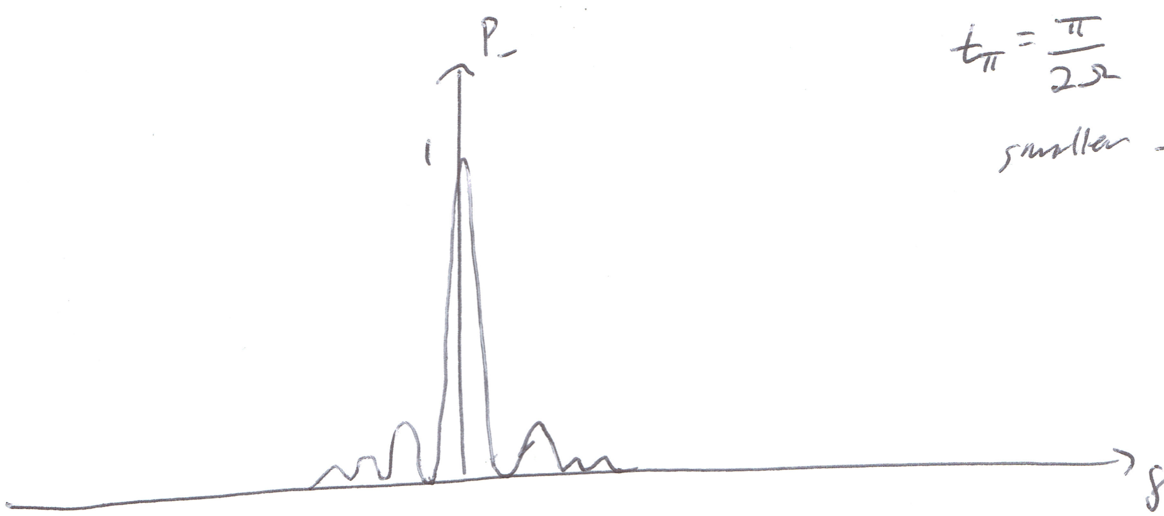
$$t_{n\pi} = n_{odd} \frac{\pi}{2\Omega}$$



For longer times, fixed pulse area $t\,\Omega = \frac{\pi}{2}$, ~~[illegible]~~ freq. spectrum gets narrower $\Delta f \Delta t \sim 1$ $\Rightarrow$ better energy resolution

$$t_{\pi} = \frac{\pi}{2\Omega}$$

smaller Ω



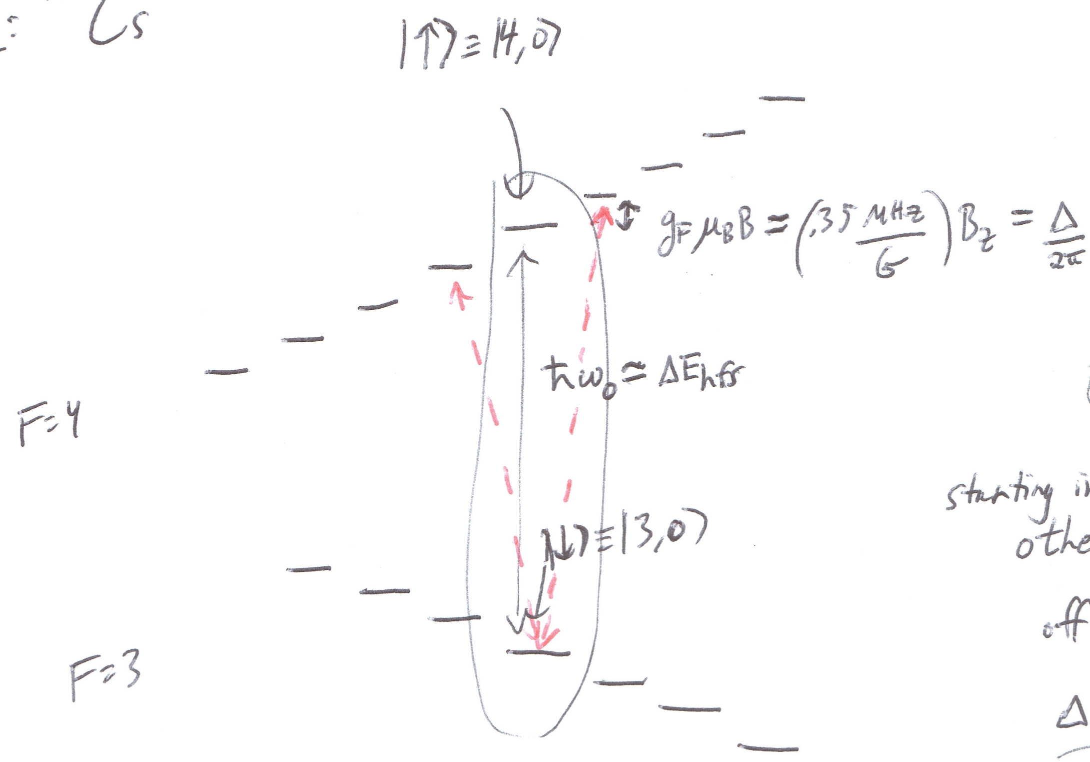
① Some important things to note

- for ~~...~~ pseudo-spin systems, where there are actually more than 2 states that may be coupled {given some conditions on radiation, selection rules

the two levels are isolated <u>spectroscopically</u>

that is: $\omega = \omega_0^{21} \equiv \dfrac{E_2 - E_1}{\hbar}$ but is off-resonant for other transition

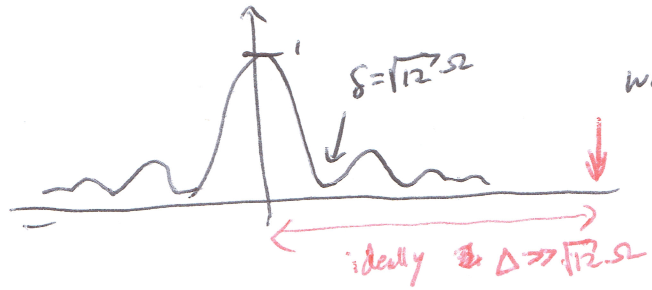
<u>ex:</u> ^{133}Cs



let's say $B_z = 2G$

starting in $|\uparrow\rangle$ or $|\downarrow\rangle$ other transitions are off-resonant by $\Delta \simeq 2\pi \times 0.7\, MHz$

To avoid off resonant excitations, want corresponding $P_{other}(t_\pi)$ to be very small (where "other" refers to states w/ dipole-allowed transitions, like $|4,\pm1\rangle$).



want $\delta^{other} \gg \sqrt{12}\,\Omega$

(9) in fact, a square / rectangle pulse is not the best for spectroscopically isolating 2 states.

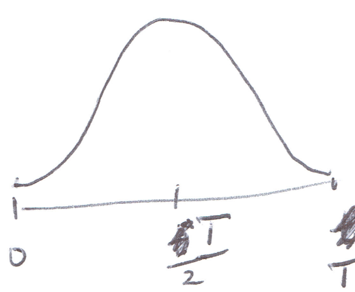
→ ~~[illegible]~~ naive improvement → to avoid higher + lower frequency components

due to Fourier-broadening, use longer pulses

→ this gets hard due to "noise", variations in ω_0

→ better improvement use a pulse shape w/ better Fourier transform, i.e. something like a Gaussian pulse

Blackman pulse w/ pulse area $= \pi/2$ $\left[\int \Omega\, dt = \pi/2\right]$ on resonance.

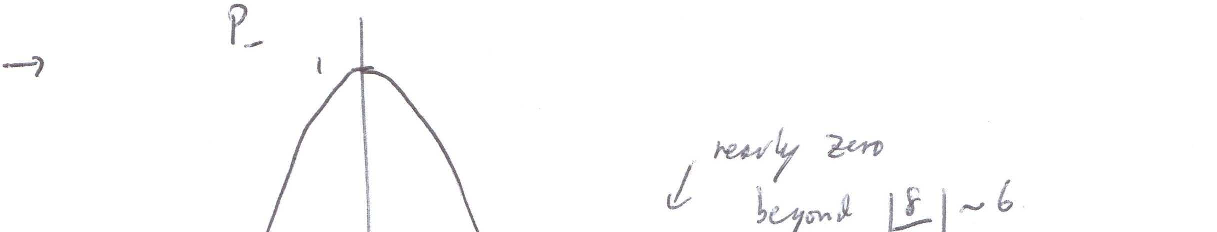


general form

$\cancel{BP(t) = \frac{21}{5} + \cos\left(\frac{\pi t}{2T}\right) + \frac{2}{25}\cos\left(\frac{2\pi t}{T}\right)}$

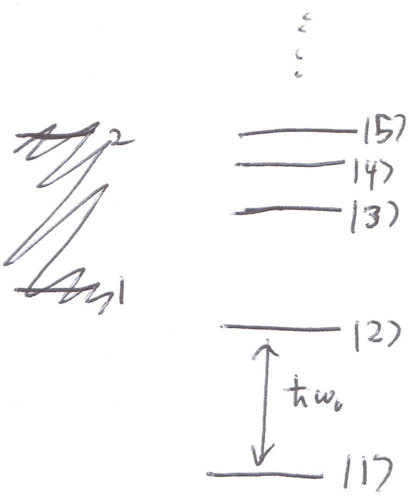
$$BP(t) = \frac{21}{5} + \frac{1}{2}\cos\left(\frac{\pi(t - T/2)}{T}\right) + \frac{2}{25}\cos\left(\frac{2\pi(t-\frac{T}{2})}{T}\right)$$

→

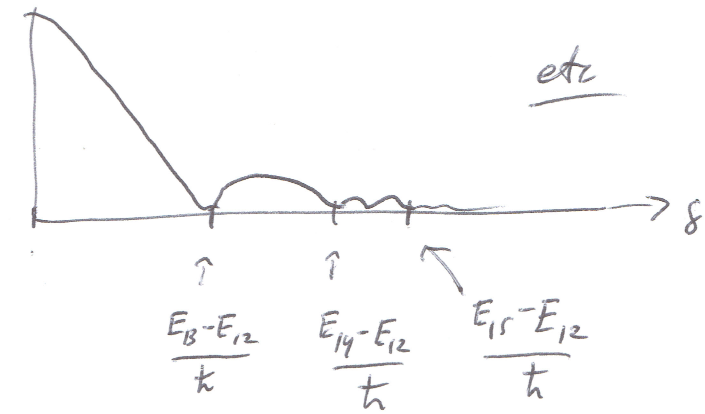


nearly zero beyond $\left|\frac{\delta}{\Omega}\right| \sim 6$

(10) in fact, for a given set of state-energies,



almost any "filter function" can be designed by controlling properties of the pulse shape



Some goals + related considerations related to applying "pulses"

1) Goal → just applying high-fidelity manipulations of this spin / pseudo spin degree of freedom

Consideration → want to be as insensitive as possible to anything that can degrade fidelity, i.e. lead to unwanted operations e.g., variations of $\hbar\omega_0 = 2\mu B_0$ for a real spin

we'll come back to this

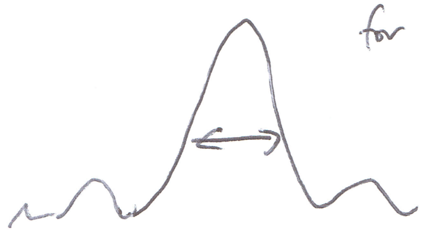
can generally be used for metrology also

let's focu on this

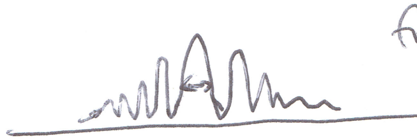
2) Goal → to measure ΔE accurately + precisely, e.g. to learn new physics (ex: Lamb shift) or to keep track of time / make an atomic clock.

Consideration → don't want oscillating field to disturb value of ΔE & want to be sensitive to variations in ΔE

so, goal is to measure small uncertainty (& inaccuracy) in ω_0



for π-pulse, $\Delta\omega_{FWHM} \sim \Omega$



for $n\pi$-pulse, $\Delta\omega_{FWHM} \sim \frac{\Omega}{n}$

improve by, longer measurement

in the end $Q \sim \frac{\omega}{\Delta\omega}$ matters

and $\Delta\omega = \frac{\Delta\omega_{FWHM}}{(S/N)}$ ← improve by
- more measurements
- improved detection
- using quantum-limited measurement (vs. shot-noise limited

for Cs atomic clock,

$\frac{\Delta\omega}{\omega}$ measured to $\sim 10^{-15}$ → for $\frac{\omega}{2\pi} \approx 9\,GHz$, this means known @ ~$\mu Hz$ level

pushing to smaller and smaller Ω, longer + longer t_π gets very hard

[many things can systematically shift ΔE, and variations in Ω, φ can be significant if Ω is very small

in fountain / beam experiments, very hard to keep Ω, φ constant over extended region of space (needed for extended measurement time)

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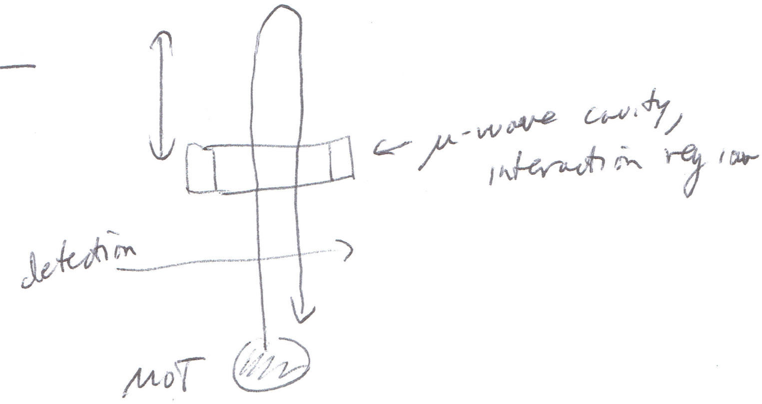
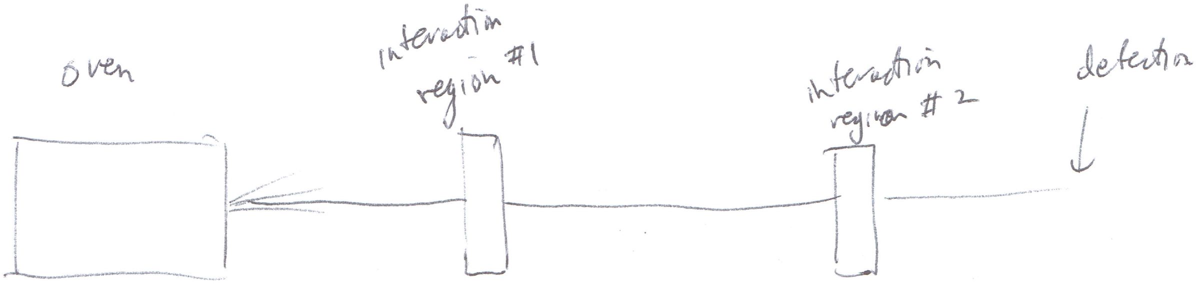
fountain



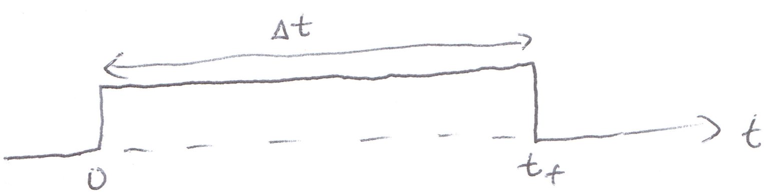
beam



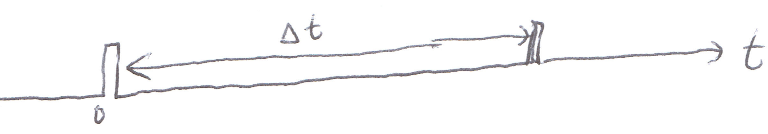
Upshot: we essentially want our "transverse field" (and really all fields except those necessary to provide ΔE) to not be present for the vast majority of our interrogation

solution by Ramsey → "method of separated oscillatory fields"

replace

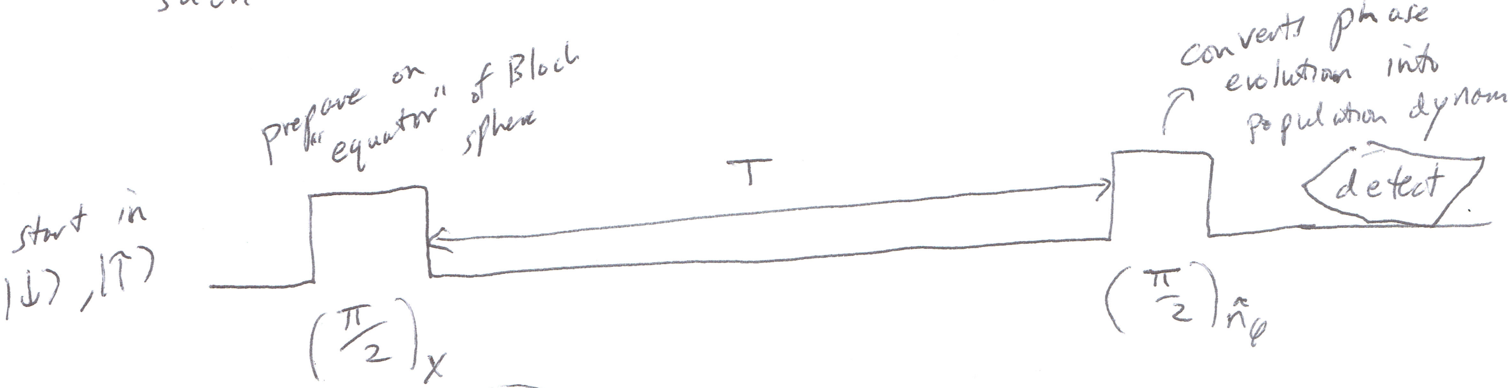


with



[free evolution in effective B_z field]

The pulses themselves can be very short (large Ω), such that even for small δ they still transfer population.



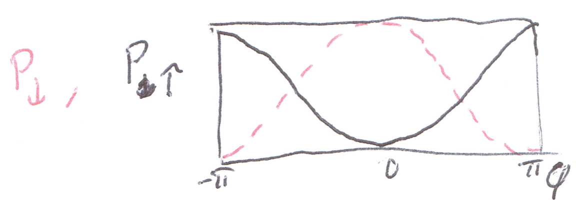
$\boxed{\text{assume } T_{pulse} \ll T}$

Let's assume $\delta = 0$ during pulses (valid if $\delta \ll \Omega$), and assume initial pulse of $\left(\frac{\pi}{2}\right)_x$ (i.e. $\varphi = 0$ during pulse #1).

→ After pulse #1, the Bloch vector for the state points along $-\hat{y}$ (assuming $\mu < 0$)

→ During long time T, Bloch vector is stationary in the rotating (G w) frame if $\delta = 0$.

→ If $\varphi = 0$ for the final $\frac{\pi}{2}$-pulse, the spin is flipped, $|\uparrow\rangle \to |\downarrow\rangle$. That is, for $\varphi = 0$, we rotate by additional $\frac{\pi}{2}$ about $\hat{x}$ spin vector. Varying φ varies the final "torque" vector, i.e. the spin vector about which we rotate. For $\varphi = \pi$, we rotate about $-\hat{x}$ ⇒ goes back to $|\uparrow\rangle$. For $\varphi = \pm\frac{\pi}{2}$, rotate about $\pm\hat{y}$ ⇒ no evolution for second pulse.



starting w/ $|\uparrow\rangle$, $\left(\frac{\pi}{2}\right)_x$ at start

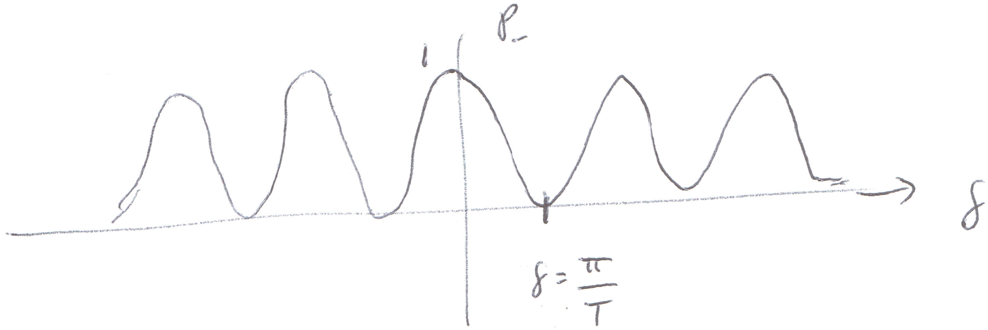
(14)

→ if $\delta \neq 0$ (but $\delta \ll \Omega$), the state initialized along $-\hat{y}$ precesses at a rate δ (in the rot. frame

→ after T, ends up rotated by angle $\Delta\phi = \delta T$

if we apply second $\left(\frac{\pi}{2}\right)$ pulse w/ $\phi = 0$

$$P_-(t) = \frac{1}{2} + \frac{1}{2}\cos(\delta T)$$



More general result (allowing for general δ, Ω) for two $\left(\frac{\pi}{2}\right)_x$ pul

$$P_- = 4\left|\frac{\Omega}{\tilde{\Omega}}\right|^2 \sin^2\left[\tilde{\Omega} t\right]\left[\cos\left(\frac{\delta T}{2}\right)\cos\left(\tilde{\Omega} t\right) - \frac{\delta}{2\tilde{\Omega}}\sin\left(\frac{\delta T}{2}\right)\sin\left(\tilde{\Omega} t\right)\right.$$

ideal result

$$P_- \approx \operatorname{sinc}^2\left(\frac{\delta \tau_p}{2}\right)\cos^2\left(\frac{\delta T}{2}\right)$$

approximating $\frac{\pi}{2}$ pulses as ideal $(\delta = 0)$

w/ duration τ_p

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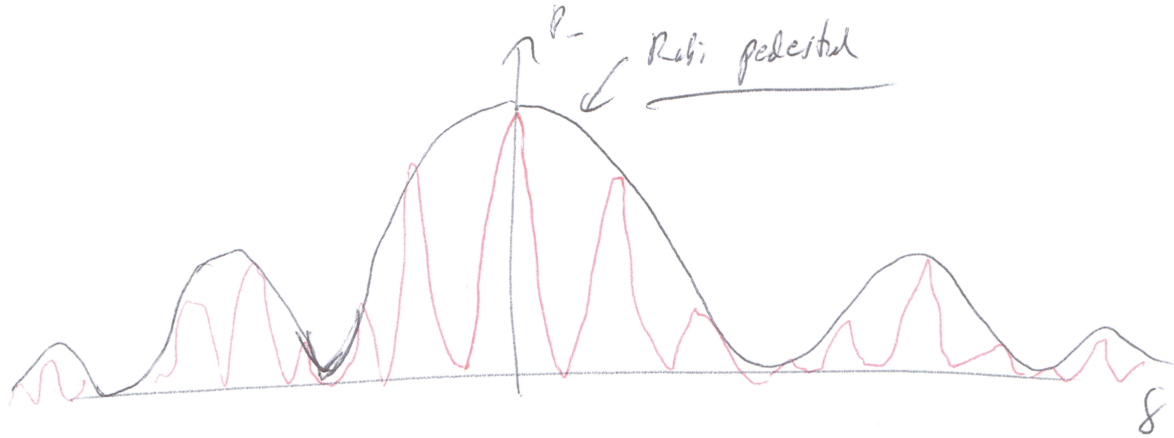
General shape



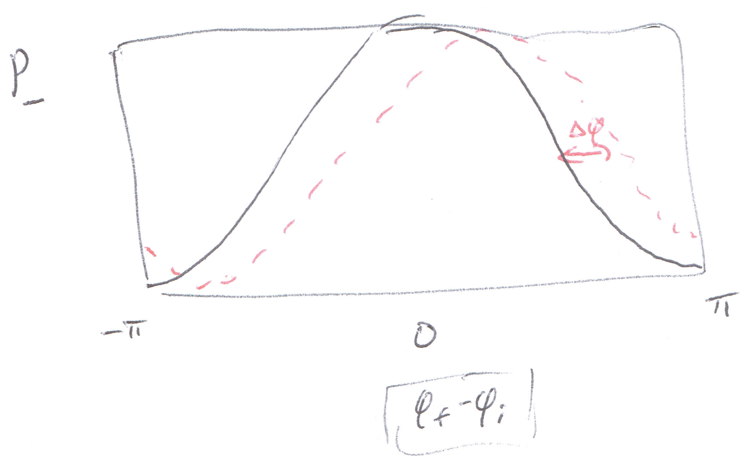
→ similar to two-slit interference → slit spacing & slit width matter

Looking back @ φ-dependence

Typically, for a fixed time, and δ, one scans φ_{final} and gets a curve

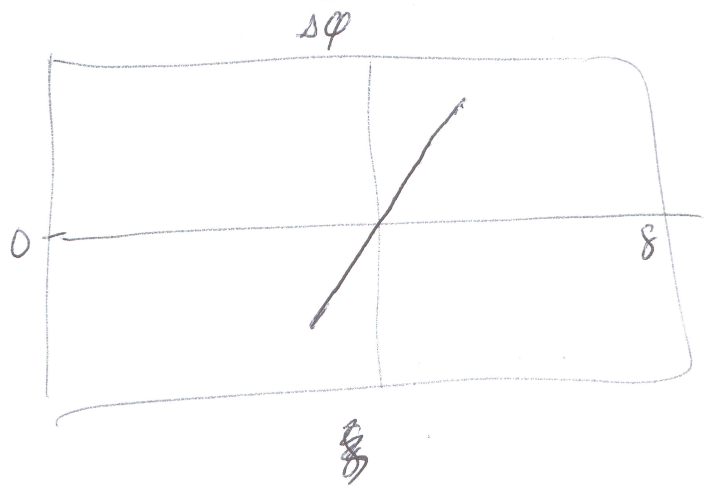
for fixed δ

$$\Delta\varphi(t) = \delta t$$



for some different value of time or δ, measure shifted curve

(16) Plotting $\frac{d}{dt}[\Delta\varphi]$ vs. δ gives



Zero-intercept gives $\delta = 0$ and $\omega = \omega_0$ condition

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Shifts ~~[illegible]~~ of ω due to applied fields, interactions between atoms, virtual photons, etc.

The Ramsey signals we showed assume <u>coherent</u> evolution.

If you have noise (classical noise on ΔE due to fluctuating fields, or classical noise on your applied oscillating field)

then the observed signal will be degraded / washed out.

Different issues due to
- time-dependence [For Cs clock, choice of states w/ $\frac{\partial(\Delta E)}{\partial B} = 0$]
- spatial variations
- certain types of interactions

can be corrected for by tailoring your pulse sequence.