PHYS 598 AQG HW2

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1. **Q1 Foot 6.2** *[1.5 pts]*

   (a) Fine and hyperfine splitting are caused by the interaction of effective magnetic field with magnetic moment of electron and nucleus respectively. Since nuclear magneton is related to Bohr magneton as:

   $$\frac{\mu_N}{\mu_B} = \frac{m_e}{M_N} \tag{1}$$

   it is apparent that hyperfine interaction is of the order $\frac{m_e}{M_N}$ smaller than fine interaction.

   (b) Consider two angular momentum operators $\vec{L}_1$, $\vec{L}_2$, $L_1 \geqslant L_2$. Define their sum as $\vec{L}_3 = \vec{L}_1 + \vec{L}_2$. Rules of addition of angular momentum dictate that the possible values of L_3 are $L_1 - L_2$, ..., $L_1 + L_2$, the total of $2L_2 + 1$ states.
   Applying this reasoning to hyperfine interaction, which is determined by the sum $\vec{F} = \vec{I} + \vec{J}$:

   i. if $J \geqslant I$, than $J \to L_1$ and $I \to L_2$ and therefore the hyperfine interaction splits a given J state onto $2I + 1$ hyperfine levels

   ii. if $I \geqslant J$, than $J \to L_2$ and $I \to L_1$ and therefore the hyperfine interaction splits a given J state onto $2J + 1$ hyperfine levels

   Using these rules we can ascribe the angular momentum indices to the levels in the Li 2s→2p transition as well as the nuclear spin to ^{6}Li and ^{7}Li:

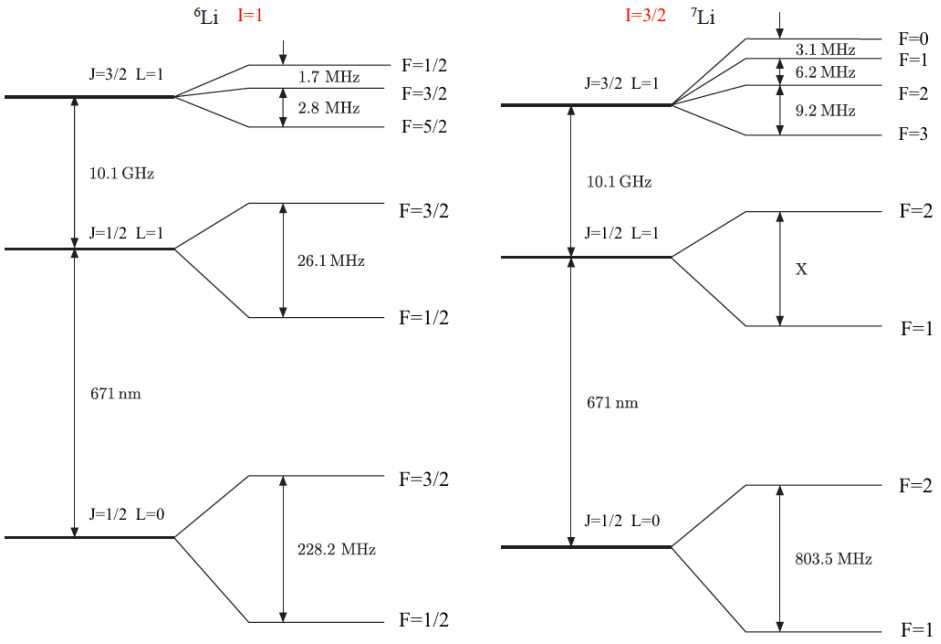

Figure 1: Li 2s→2p transition.

Note the order of increasing/decreasing of F-terms within each J-multiplet. We knew to assign the higher value F to higher energy for $J = 1/2$, $L = 0$ and $J = 1/2$, $L = 1$ since for them $A > 0$. In contrast, this order is reversed for $J = 3/2$, $L = 1$, which is apparent from the interval rule. In particular, we know that for increasing F the energy gaps in the multiplet should increase.
To evaluate the value of X we can once again use the interval rule:

$$E_F - E_{F-1} = AF \tag{2}$$

We can now apply the rule to $J = 1/2$, $L = 0$ and $J = 1/2$, $L = 1$ multiplets of ^{6}Li and ^{7}Li. Here we need to make an important assumption, which is that the ratio of the $P_{1/2}$ hyperfine splitting to the $S_{1/2}$ hyperfine splitting is independent of the isotope, i.e. whether an extra neutron is included in the nucleus. That is, this ratio comes about due to the different amounts of nuclear charge screening experienced by the electron (due to inner shell electrons) depending on whether it is in the s or p orbital. While A may change from isotope to isotope, this ratio should be essentially independent.

$$\begin{aligned}\frac{E_6(J=1/2,L=1,F=3/2)-E_6(J=1/2,L=1,F=1/2)}{E_6(J=1/2,L=0,F=3/2)-E_6(J=1/2,L=0,F=1/2)} = \frac{A(J=1/2,L=1)}{A(J=1/2,L=0)} = \\ = \frac{E_7(J=1/2,L=1,F=3/2)-E_7(J=1/2,L=1,F=1/2)}{E_7(J=1/2,L=0,F=3/2)-E_7(J=1/2,L=0,F=1/2)}\end{aligned} \tag{3}$$

$$\frac{228.2}{803.5} = \frac{26.1}{X} \tag{4}$$

$$X = 91.9\text{MHz} \tag{5}$$

2. **Q2 Atom in a magnetic field** *[5.5 pts]*

$$H = \frac{hA_{hfs}}{\hbar^2}\vec{I}\cdot\vec{J} + \frac{\mu_B}{\hbar}\Big(g_j J_z + g_I I_z\Big)B_z \tag{6}$$

(a) It will be convenient to work in the basis $|m_I, m_J\rangle$. To this end we bring the Hamiltonian to an appropriate form:

$$H = \frac{hA_{hfs}}{\hbar^2}I_z J_z + \frac{hA_{hfs}}{2\hbar^2}\Big(J_+ I_- + J_- I_+\Big) + \frac{\mu_B}{\hbar}\Big(g_J J_z + g_I I_z\Big)B_z \tag{7}$$

Note that in this basis the only non-zero matrix elements of the Hamiltonian are diagonal elements and coupling elements between states $|m_I = m + 1/2, m_J = -1/2\rangle$ and $|m_I = m - 1/2, m_J = 1/2\rangle$. Thus the matrix splits into 3 blocks: $H_{block} =$

$$\begin{bmatrix} \mu_B B_z(mg_I + \frac{1}{2}(g_J - g_I)) + \frac{hA_{hfs}}{2}(m - \frac{1}{2}) & \frac{hA_{hfs}}{2}\sqrt{(I+\frac{1}{2})^2 - m^2} \\ \frac{hA_{hfs}}{2}\sqrt{(I+\frac{1}{2})^2 - m^2} & \mu_B B_z(mg_I - \frac{1}{2}(g_J - g_I)) - \frac{hA_{hfs}}{2}(m + \frac{1}{2}) \end{bmatrix} \tag{8}$$

and 2 more diagonal elements. The latter correspond to the states $|m_I = 3/2, m_J = 1/2\rangle$ and $|m_I = -3/2/2, m_J = -1/2\rangle$ that do not couple to others. We can thus diagonalize the whole Hamiltonian by diagonalizing each separate block. Proceeding this way we arrive at the Breit-Rabi formula:

$$E^{\pm} = -\frac{\Delta E_{hfs}}{4F^+} + g_I m\mu_B B_z \pm \frac{\Delta E_{hfs}}{2}\sqrt{1 + \frac{4mx}{2F^+} + x^2} \tag{9}$$

where $F^+ = I + \frac{1}{2}$, $\frac{\Delta E_{hfs}}{4} = hA_{hfs}F^+$, $x = \frac{\mu_B B_z (g_J - g_I)}{\Delta E_{hfs}}$. These eigenvalues correspond to the eigenvectors:

$$|+\rangle = \frac{\alpha_1}{\sqrt{\alpha_1^2 + \alpha_2^2}} |m + 1/2, -1/2\rangle + \frac{\alpha_2}{\sqrt{\alpha_1^2 + \alpha_2^2}} |m - 1/2, 1/2\rangle \tag{10}$$

$$|-\rangle = \frac{\beta_1}{\sqrt{\beta_1^2 + \beta_2^2}} |m + 1/2, -1/2\rangle + \frac{\beta_2}{\sqrt{\beta_1^2 + \beta_2^2}} |m - 1/2, 1/2\rangle \tag{11}$$

with coefficients $\alpha_1 = \beta_1 = \sqrt{1 - \frac{m^2}{(F^+)^2}}$, $\alpha_2 = \sqrt{1 + \frac{4mx}{2F^+} + x^2} - (x + \frac{m}{F^+})$, $\beta_2 = -\sqrt{1 + \frac{4mx}{2F^+} + x^2} - (x + \frac{m}{F^+})$.

And the energies of $|m_I = 3/2, m_J = 1/2\rangle$ and $|m_I = -3/2/2, m_J = -1/2\rangle$ are:

$$E(m_I = \pm 3/2, m_J = \pm 1/2) = \frac{3hA_{hfs}}{4} \pm \mu_B \Big(\frac{1}{2} g_J + \frac{3}{2} g_I\Big) B_z \tag{12}$$

(b) Plotting the energy dependencies of the 8 states that exactly diagonalize the Hamiltonian as a function of magnetic field strength:

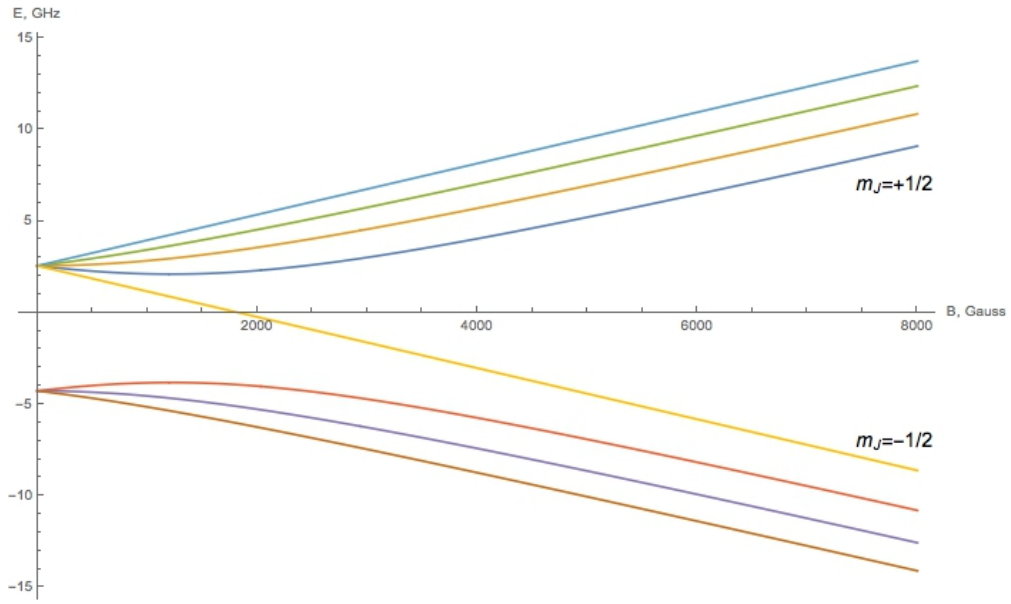

Figure 2: Hyperfine structure and Zeeman splitting of 8 levels in the $S_{1/2}$ multiplet in magnetic field.

At low magnetic field the 8 states split into 5 states with $F = 2$ and 3 states with $F = 1$. At higher magnetic fields the states regroup into pair of 4 states with $m_J = \pm 1/2$.

(c)

$$\Delta E = E(F = 2, m_F = -1) - E(F = 1, m_F = -1) = \Delta E_{hfs} \sqrt{1 + \frac{4mx}{2F^+} + x^2} \tag{13}$$

The insensitivity to magnetic field will occur when $\frac{\partial \Delta E}{\partial B} = \frac{\partial \Delta E}{\partial x} = 0$.

$$\frac{\partial \Delta E}{\partial x} = 0 \;\Rightarrow\; x = \frac{1}{I + 1/2} = \frac{1}{2} \tag{14}$$

which corresponds to magnetic field:

$$B = \frac{\Delta E_{hfs}}{F^+ (g_J - g_I) \mu_B} = 1220 \text{ Gauss} \tag{15}$$

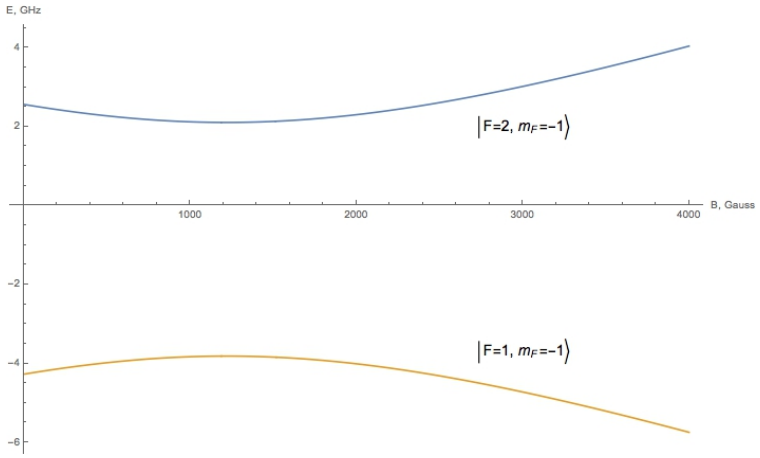

Figure 3: Energy of $|F = 2, m_F = -1\rangle$ and $|F = 1, m_F = -1\rangle$ as a function of the magnetic field strength.

At this magnetic field the states $|F = 2, m_F = -1\rangle$, $|F = 1, m_F = -1\rangle$ in the $|m_I, m_j\rangle$ basis are represented by:

$$|F = 2, m_F = -1\rangle = \frac{1}{\sqrt{2}}\Big(|-3/2, 1/2\rangle + |-1/2, -1/2\rangle \Big) \tag{16}$$

$$|F = 1, m_F = -1\rangle = \frac{1}{\sqrt{2}}\Big(|-3/2, 1/2\rangle - |-1/2, -1/2\rangle \Big) \tag{17}$$

Note that at this particular value of magnetic field the electronic contribution to the magnetic moment is zero, which causes the insensitivity to magnetic field.

3. **Q3 Foot 1.8** *[1.5 pts]*

Power radiated by a classical harmonic oscillator is:

$$P = \frac{e^2 r^2 w^4}{12\pi\epsilon_0 c^3} \tag{18}$$

We can think of an electron on a circular orbit as 2D harmonic oscillator, and thus its radiation power is:

$$P = \frac{e^2 r^2 w^4}{6\pi\epsilon_0 c^3} \tag{19}$$

The angular frequency of oscillation is:

$$\omega = \frac{2\pi c}{\lambda} = 2.87 \cdot 10^{15} (\text{rad/s}) \tag{20}$$

Then the time to radiate energy that corresponds to one photon, $E = \hbar\omega$, is:

$$t = \frac{\hbar\omega}{P} = \frac{6\hbar\pi\epsilon_0 c^3}{e^2 r^2 \omega^3} \tag{21}$$

The transition under consideration is $n = 3 \to n = 2$. We thus take r to be equal to the radius of the third Bohr orbit: $r = 3^2 \times a_0$ and arrive at the time $t = 3.1 \cdot 10^{-8}$ s.

4. **Q3 Foot 7.2** *[1.5 pts]*

(a)

$$i\dot{c}_1 = c_2 e^{i(\omega-\omega_0)t}\frac{\Omega}{2} \tag{22}$$

$$i\dot{c}_2 = c_1 e^{i(\omega-\omega_0)t}\frac{\Omega^*}{2} \;\Rightarrow\; c_1 = \frac{2i}{\Omega^*}e^{-i(\omega-\omega_0)t}\dot{c}_2 \tag{23}$$

$$i\dot{c}_1 = c_2 e^{i(\omega-\omega_0)t}\frac{\Omega}{2} = \frac{2i}{\Omega^*}e^{-i(\omega-\omega_0)t}\ddot{c}_2 + \frac{2(\omega-\omega_0)}{\Omega^*}\dot{c}_2 \tag{24}$$

and thus:

$$\ddot{c}_2 + i(\omega-\omega_0)\dot{c}_2 + \left|\frac{\Omega}{2}\right|^2 c_2 = 0 \tag{25}$$

Upon change of variables:

$$x = c_2 e^{\frac{i}{2}(\omega-\omega_0)t} \tag{26}$$

the Eq.(25) becomes:

$$\ddot{x} + \frac{1}{4}\Big(|\Omega|^2 + (\omega-\omega_0)^2\Big)x = 0 \tag{27}$$

Using the initial condition $c_2(t=0) = 0 = x(t=0)$ we arrive at the solution:

$$x = B\,\sin(\frac{1}{2}\sqrt{|\Omega|^2 + (\omega-\omega_0)^2}t) \tag{28}$$

or

$$c_2 = B\,\sin(\frac{1}{2}\sqrt{|\Omega|^2 + (\omega-\omega_0)^2}t)e^{-\frac{i}{2}(\omega-\omega_0)t} \tag{29}$$

which is indeed consistent with (7.27). We can also deduce the constant B from the boundary condition $c_1(0) = 1$. Substituting the Eq.(29) into Eq.(23) and using the boundary condition we obtain:

$$B = \frac{\Omega^*}{i\sqrt{|\Omega|^2 + (\omega-\omega_0)^2}} \tag{30}$$

(b) Ploting the $|c_2(t)|^2$:

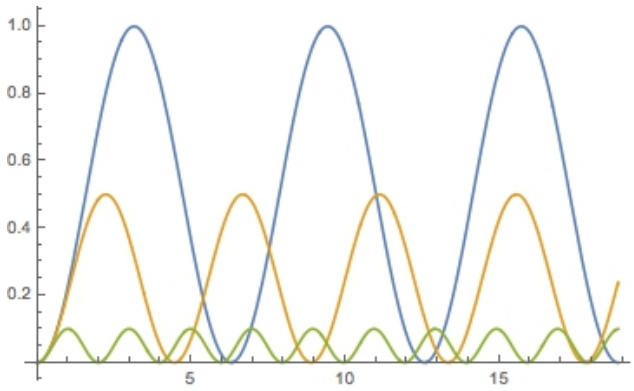

Figure 4: