# PHYS 598 AQG HW3

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1. **Q1** *[7 pts; 1 pt per part]*

   (a) Dynamics of the state:

$$|\psi\rangle = \begin{bmatrix} \langle e|\psi\rangle \\ \langle g|\psi\rangle \end{bmatrix} = \begin{bmatrix} a_e \\ a_g \end{bmatrix} \tag{1}$$

is governed by:

$$i\hbar\partial_t |\psi\rangle = \mathcal{H} |\psi\rangle \tag{2}$$

where

$$\mathcal{H} = \begin{bmatrix} \frac{\hbar\omega_0}{2} & -h_\perp(t)e^{-i(\omega t+\phi(t))} \\ -h_\perp(t)e^{i(\omega t+\phi(t))} & \frac{\hbar\omega_0}{2} \end{bmatrix} \tag{3}$$

Writing Eq. 2 in components:

$$i\hbar\dot{a_e} = \frac{\hbar\omega_0}{2}a_e - a_g h_\perp(t)e^{-i(\omega t+\phi(t))} \tag{4}$$

$$i\hbar\dot{a_g} = -\frac{\hbar\omega_0}{2}a_g - a_e h_\perp(t)e^{i(\omega t+\phi(t))} \tag{5}$$

   (b) Now we move to a rotating frame:

$$\begin{bmatrix} b_e \\ b_g \end{bmatrix} = \begin{bmatrix} a_e e^{\frac{i\omega t}{2}} \\ a_g e^{-\frac{i\omega t}{2}} \end{bmatrix} \tag{6}$$

where dynamics of the state is given by:

$$i\hbar\partial_t \begin{bmatrix} b_e \\ b_g \end{bmatrix} = \begin{bmatrix} -\frac{\hbar\delta}{2} & -h_\perp e^{-i\phi} \\ -h_\perp e^{i\phi} & \frac{\hbar\delta}{2} \end{bmatrix} \begin{bmatrix} b_e \\ b_g \end{bmatrix} = \mathcal{H}_b |\psi\rangle_b \tag{7}$$

Although in this basis $\mathcal{H}$ is not diagonal, $\mathcal{H}^2$ is:

$$-\hbar^2\partial_t^2 = \mathcal{H}_b^2 |\psi\rangle_b = \begin{bmatrix} h_\perp^2 + \frac{\delta^2\hbar^2}{4} & 0 \\ 0 & h_\perp^2 + \frac{\delta^2\hbar^2}{4} \end{bmatrix} \begin{bmatrix} b_e \\ b_g \end{bmatrix} \tag{8}$$

Using the B.C.:

$$\begin{bmatrix} a_e \\ a_g \end{bmatrix} (t=0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{9}$$

or equivalantly:

$$\begin{bmatrix} b_e \\ b_g \end{bmatrix} (t=0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{10}$$

and Eq. 7 we arrive at a set of B.C. that uniquely defines the solution of Eq. 8:

$$\begin{bmatrix} b_e \\ b_g \end{bmatrix} (t=0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \partial_t \begin{bmatrix} b_e \\ b_g \end{bmatrix} (t=0) = \begin{bmatrix} \frac{i\delta}{2} \\ i\frac{h_\perp}{\hbar} e^{i\phi} \end{bmatrix} \tag{11}$$

Such a solution is:

$$\begin{bmatrix} b_e \\ b_g \end{bmatrix} = \begin{bmatrix} \cos(\alpha t) + \frac{i\delta}{2\alpha}\sin(\alpha t) \\ ie^{i\phi}\frac{h_\perp}{\hbar\alpha}\sin(\alpha t) \end{bmatrix} \tag{12}$$

where we defined $\alpha = \sqrt{\frac{h_\perp^2}{\hbar^2} + \frac{\delta^2}{4}}$.

We are now ready to calculate the probability $P_g(t)$:

$$P_g(t) = \left|b_g(t)\right|^2 = \frac{h_\perp^2}{\hbar^2\alpha^2}\sin^2\Big(\alpha t\Big) \tag{13}$$

Taking $0.1\omega_0 = \frac{h_\perp}{\hbar}$ and plotting $P_g(t)$ for:

i. $\delta = 0$, $P_g(t) = \sin^2(\frac{\omega_0}{10}t)$

ii. $\delta = \frac{h_\perp}{\hbar}$, $P_g(t) = \frac{4}{5}\sin^2(\frac{\omega_0}{10}\sqrt{\frac{5}{4}}t)$

iii. $\delta = \frac{10h_\perp}{\hbar}$, $P_g(t) = \frac{1}{26}\sin^2(\frac{\omega_0}{10}\sqrt{26}t)$

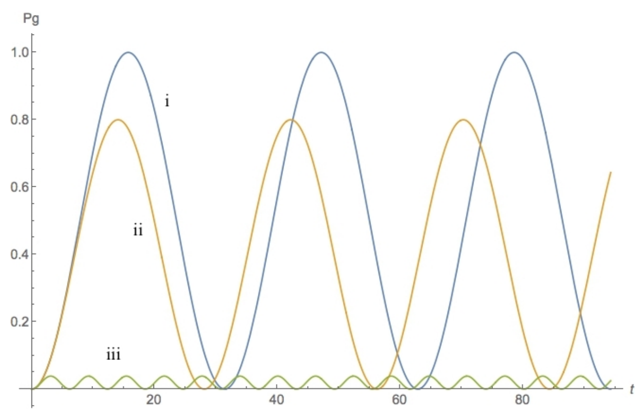


Figure 1: $P_g(t)$ for $\delta = 0$, $\delta = \frac{h_\perp}{\hbar}$ and $\delta = \frac{10h_\perp}{\hbar}$ .

(c) The initial state $|\psi\rangle = \frac{1}{\sqrt{2}}|g\rangle - \frac{i}{\sqrt{2}}|e\rangle$ corresponds to the B.C.:

$$\begin{bmatrix} b_e \\ b_g \end{bmatrix} (t=0) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix} \qquad \partial_t \begin{bmatrix} b_e \\ b_g \end{bmatrix} (t=0) = \frac{h_\perp}{\sqrt{2}\hbar} \begin{bmatrix} ie^{-i\phi} \\ e^{i\phi} \end{bmatrix} \tag{14}$$

which corresponds to the solution of Eq. 7:

$$\begin{bmatrix} b_e \\ b_g \end{bmatrix} (t) = \frac{1}{\sqrt{2}} \begin{bmatrix} -i\cos\big(\frac{h_\perp t}{\hbar}\big) + i\sin\big(\frac{h_\perp t}{\hbar}\big)e^{-i\phi} \\ \cos\big(\frac{h_\perp t}{\hbar}\big) + \sin\big(\frac{h_\perp t}{\hbar}\big)e^{i\phi} \end{bmatrix} \tag{15}$$

Using this solution and assuming $0.1\omega_0 = \frac{h_\perp}{\hbar}$ we arrive at:

i. for $\phi = 0$, $P_g(t) = \left|b_g(t)\right|^2 = \frac{1}{2}\Big(1 + \sin\big(\frac{\omega_0 t}{5}\big)\Big)$.

ii. for $\phi = \frac{\pi}{2}$, $P_g(t) = \frac{1}{2}$.

iii. for $\phi = \pi$, $P_g(t) = \left|b_g(t)\right|^2 = \frac{1}{2}\Big(1 - \sin\big(\frac{\omega_0 t}{5}\big)\Big)$.

iv. for $\phi = \frac{3\pi}{2}$, $P_g(t) = \frac{1}{2}$.

Plotting these probabilities:

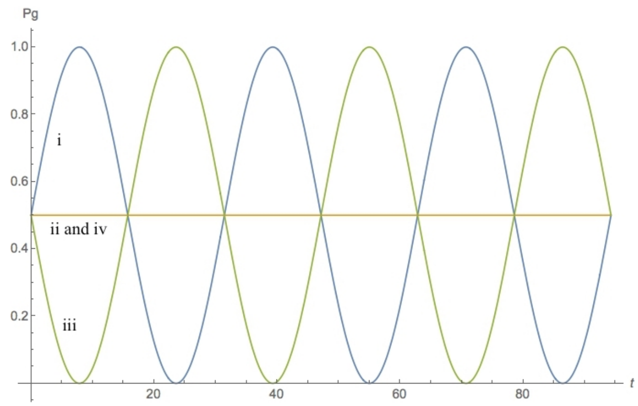


Figure 2: $P_g(t)$ for $\phi = 0$, $\phi = \frac{\pi}{2}$, $\phi = \pi$, $\phi = \frac{3\pi}{2}$.

(d) We can rewrite the solutions for the four cases considered in part c as:

i. $|\psi\rangle_b = -\frac{i}{\sqrt{2}}\left[\left(\cos(\frac{h_\perp t}{\hbar}) - \sin(\frac{h_\perp t}{\hbar})\right)|e\rangle + \left(\cos(\frac{h_\perp t}{\hbar}) + \sin(\frac{h_\perp t}{\hbar})\right)|g\rangle\right]$

ii. $|\psi\rangle_b = -\frac{i}{\sqrt{2}}e^{\frac{ih_\perp t}{\hbar}}\left[|e\rangle + e^{\frac{i\pi}{2}}e^{\frac{-i2h_\perp t}{\hbar}}|g\rangle\right]$

iii. $|\psi\rangle_b = -\frac{i}{\sqrt{2}}\left[\left(\cos(\frac{h_\perp t}{\hbar}) + \sin(\frac{h_\perp t}{\hbar})\right)|e\rangle + \left(\cos(\frac{h_\perp t}{\hbar}) - \sin(\frac{h_\perp t}{\hbar})\right)|g\rangle\right]$

iv. $|\psi\rangle_b = -\frac{i}{\sqrt{2}}e^{\frac{ih_\perp t}{\hbar}}\left[|e\rangle + e^{\frac{i\pi}{2}}e^{\frac{i2h_\perp t}{\hbar}}|g\rangle\right]$

Now we note that a general solution on Bloch sphere can be expressed as:

$$|\psi\rangle = e^{i[overall \;\; phase]}\left[\cos(\frac{\theta}{2})|+\rangle + \sin(\frac{\theta}{2})e^{i\phi}|-\rangle\right] \tag{16}$$

where $\theta$ and $\phi$ have the conventional meaning of spherical coordinates. It follows than that our four cases can be interpreted as:

i. rotation in the zx-plane around y-axis;
ii. rotation in the xy-plane around z-axis;
iii. rotation in the zx-plane around y-axis;
iv. rotation in the xy-plane around z-axis;

Note that the rotation in cases i and iii happens around the same axis but in the opposite directions. The same is true about the cases ii and iv.

(e) We will compare dynamics of the initial state from part c governed by $\mathcal{H}$ and $\mathcal{H}'$, where:

$$\mathcal{H}' = \begin{bmatrix} \frac{\hbar\omega_0}{2} & -2h_\perp(t)\cos(\omega t + \phi) \\ -2h_\perp(t)\cos(\omega t + \phi) & \frac{\hbar\omega_0}{2} \end{bmatrix} \tag{17}$$

Solving for the dynamics numerically, we plot the $P_g(t)$ for the two coupling strengths:

i. $h_\perp = 0.1\hbar\omega_0$

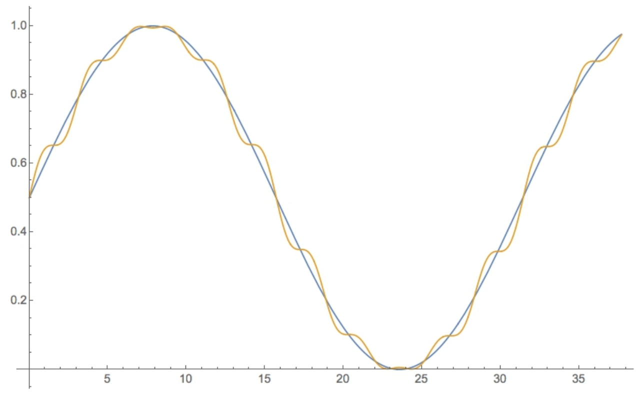


Figure 3: Comparison of $P_g(t)$ for $\mathcal{H}$ and $\mathcal{H}'$ with $h_\perp = 0.1\hbar\omega_0$.

ii. $h_\perp = 0.01\hbar\omega_0$

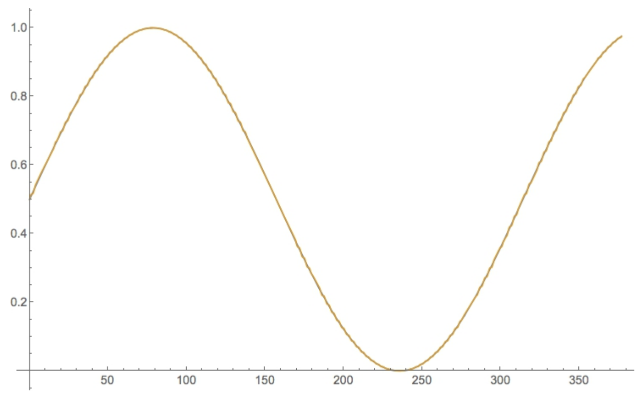


Figure 4: Comparison of $P_g(t)$ for $\mathcal{H}$ and $\mathcal{H}'$ with $h_\perp = 0.01\hbar\omega_0$.

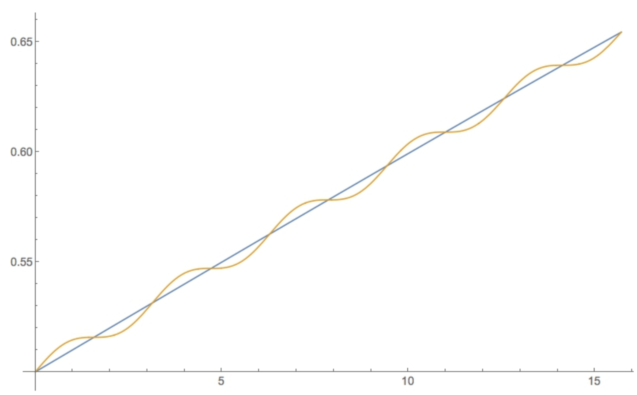


Figure 5: Comparison of $P_g(t)$ for $\mathcal{H}$ and $\mathcal{H}'$ with $h_\perp = 0.01\hbar\omega_0$ (closer view).

The main distinction between evolution under $\mathcal{H}$ and $\mathcal{H}'$ is an additional harmonic modulation in case of $\mathcal{H}'$ in comparison to $\mathcal{H}$. This modulation can be understood in terms of rotating-frame approximation. To this end we look at $\mathcal{H}'$ in a rotating frame:

$$\mathcal{H}'_{rot.r.f.} = \begin{bmatrix} \frac{\hbar\omega_0}{2} & -h_\perp(t)\big(e^{i\omega t}+e^{-i\omega t}\big)e^{i\omega t} \\ -h_\perp(t)\big(e^{i\omega t}+e^{-i\omega t}\big)e^{-i\omega t} & \frac{\hbar\omega_0}{2} \end{bmatrix} \tag{18}$$

$$\mathcal{H}'_{rot.r.f.} = \begin{bmatrix} \frac{\hbar\omega_0}{2} & -h_\perp(t) - h_\perp(t)e^{2i\omega t} \\ h_\perp(t) - h_\perp(t)e^{-2i\omega t} & \frac{\hbar\omega_0}{2} \end{bmatrix} \tag{19}$$

If we neglect the highly oscillating terms proportional to $e^{2i\omega t}$, $\mathcal{H}_b = \mathcal{H}'_{rot.fr.}$, but taking these terms into account results in harmonic modulation that we witnessed in our numerical solution.

(f) Using the solution Eq. 12 and requiring that it takes the form: $\frac{1}{\sqrt{2}}(-i\,|e\rangle + |g\rangle)$ on resonance:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix} = e^{i\chi}\begin{bmatrix} \cos(\alpha\tau_{\pi/2}) \\ i\frac{h_\perp}{\hbar\alpha}\sin(\alpha\tau_{\pi/2}) \end{bmatrix} \tag{20}$$

we obtain:

$$\alpha\tau_{\pi/2} = \pi/4, \quad \alpha = \frac{h_\perp}{\hbar} \quad \text{for } \chi = -\pi/2 \tag{21}$$

Dynamics of this new state is given by Eq. 15, which for $\alpha\tau_{\pi/2} = \pi/4$ results in:

$$\begin{bmatrix} b_e \\ b_g \end{bmatrix}(t) = \frac{1}{2}\begin{bmatrix} -i(1-e^{-i\phi}) \\ 1+e^{i\phi} \end{bmatrix} \tag{22}$$

And thus the population of $|g\rangle$ as a function of $\psi$ after $\tau_{\pi/2} = \frac{\pi}{4\alpha}$ is:

$$P_g = \left|b_g\right|^2 = \frac{1}{2}\big(1+\cos(\phi)\big) \tag{23}$$

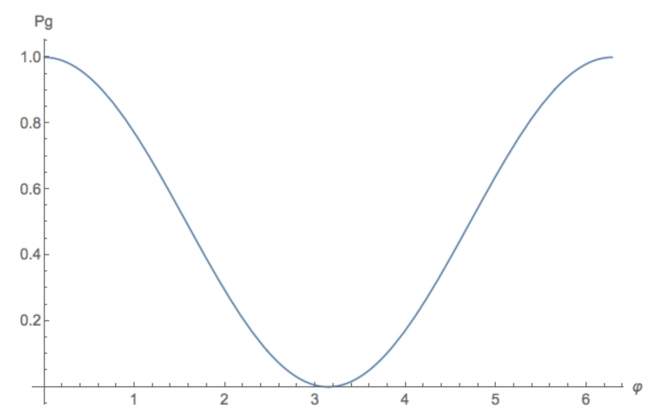


Figure 6: Population of $|g\rangle$ as a function of $\phi$ after a pulse $\tau_{\pi/2} = \frac{\pi}{4\alpha}$.

(g) We can study the population $P_g$ after for the sequence of $\pi/2$ pulse, intervening time and the second $\pi/2$ pulse numerically. To this end we use Mathematica to solve Eq. 4 and 5 for this sequence. For concreteness we take $\hbar\omega = 1$, $h_\perp = 0.1$, then plotting the resulting dependence of $P_g$ on $\phi$ (of the second $\pi/2$ pulse) for several values of $h'_\perp$:

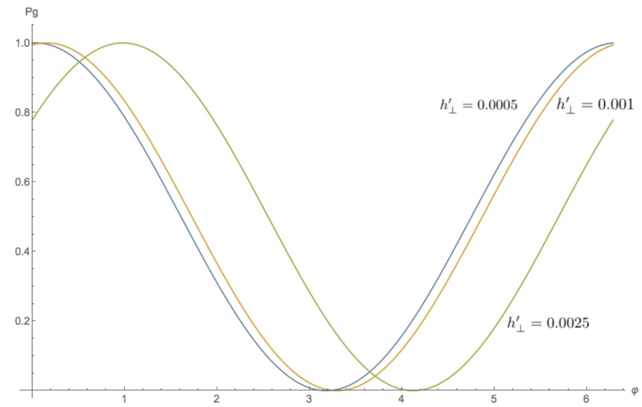


Figure 7: Population of $|g\rangle$ as a function of $\phi$ for $h'_\perp = 0.0005$, $h'_\perp = 0.001$, $h'_\perp = 0.0025$.

We notice that for small enough values of $h'_\perp$ the dependence converges to the one with no intervening pulse, but around $h'_\perp = 0.01 h_\perp$ the curve starts to shift away while preserving the shape. The "phase shift" of this curve away from the $h'_\perp = 0$ curve appears to scale superlinearly with $h'_\perp$, and in fact scales like $(h'_\perp)^2$.
Performing the same procedure, but now for the dependence of $P_g$ on $\phi'$ rather than $\phi$, we plot it for the same values of $h'_\perp$:

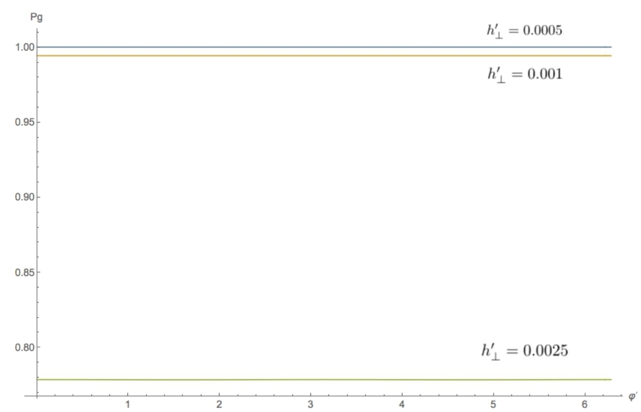


Figure 8: Population of $|g\rangle$ as a function of $\phi'$ for $h'_\perp = 0.0005$, $h'_\perp = 0.001$, $h'_\perp = 0.0025$.

As apparent from the plot, there is no dependence of $P_g$ on $\phi'$. Both of these observations are consistent with the "phase shift" being related to AC Stark shift of the ground and excited states by the far-off-resonant field. The very long time of the Ramsey measurement allows even very small energy shifts to be measurable.

2. **Q2 Foot 7.4** *[3 pts]*
   (a) In case of $\Omega = 0$ Eq. 7.95 becomes:

$$\dot{c_2} = -\frac{\Gamma}{2} c_2 \tag{24}$$

solution of which is:

$$c_2(t) = c_2(0) e^{-\frac{\Gamma t}{2}} \tag{25}$$

and thus:

$$\left|c_2(t)\right|^2 = \left|c_2(0)\right|^2 e^{-\Gamma t} \tag{26}$$

(b) Integrating the equation:

$$\frac{d}{dt}\left(c_1 e^{\frac{\Gamma t}{2}}\right) = -ic_2\frac{\Omega^*}{2}e^{-i(\omega-\omega_0+\frac{i\Gamma}{2})t} \tag{27}$$

we obtain:

$$c_2(t)e^{\frac{\Gamma t}{2}} = -i\frac{\Omega^*}{2}\int_0^t dt' c_1 e^{-i(\omega-\omega_0+\frac{i\Gamma}{2})t'} \simeq -i\frac{\Omega^*}{2}\int_0^t dt' e^{-i(\omega-\omega_0+\frac{i\Gamma}{2})t'} \tag{28}$$

$$c_2(t) = e^{-\frac{\Gamma t}{2}}\frac{\Omega^*(e^{\frac{\Gamma t}{2}}e^{-i(\omega-\omega_0)}-1)}{2(\omega-\omega_0+i\Gamma/2)} \tag{29}$$

After a time which is long compared to the radiative lifetime:

$$c_2(t) = \frac{\Omega^* e^{-i(\omega-\omega_0)}}{2(\omega-\omega_0+i\Gamma/2)} \tag{30}$$

$$\left|c_2(t)\right|^2 = \frac{\Omega^*\Omega}{4}\frac{1}{(\omega-\omega_0+i\Gamma/2)(\omega-\omega_0-i\Gamma/2)} = \frac{\left|\Omega\right|^2}{4(\omega-\omega_0)^2+\Gamma^2} \tag{31}$$