PHYS 598 AQG HW4

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1. **Q1 Foot 7.5** *[3 pts]*

   (a) Attenuation of intensity of a weak beam is described by:

   $$\frac{dI}{dz} = -N\sigma(\omega)I \;\Rightarrow\; I = I_0 e^{-N\sigma z} \tag{1}$$

   $$\sigma(\omega) = \frac{g_2}{g_1}\frac{\pi^2 c^2}{\omega_0^2} A_{21} g_H(\omega) \tag{2}$$

   $$g_H = \frac{1}{2\pi}\frac{\Gamma}{(\omega-\omega_0)^2 + \Gamma^2/4} \tag{3}$$

   For a resonant beam $\omega = \omega_0$:

   $$g_H(\omega) = \frac{2}{\pi\Gamma} \tag{4}$$

   and the two-level process implies:

   $$\Gamma = A_{21} \tag{5}$$

   $$\sigma(\omega_0) = \frac{6}{\pi}\frac{\pi^2 c^2}{\omega_0^2} = \frac{3}{2}\frac{\lambda_0^2}{\pi} \tag{6}$$

   To provide the transmission of e^{-1}:

   $$N\sigma(\omega)z = 1 \;\Rightarrow\; N = \frac{2}{3}\frac{\pi}{z\lambda_0^2} \tag{7}$$

   And the number of atoms:

   $$\mathcal{N} = VN = 3.16 \cdot 10^6 \tag{8}$$

   (b) The absorption coefficient is given by:

   $$\kappa(\omega, I) = \frac{N\sigma(\omega)}{1 + I/I_{sat}(\omega)} \tag{9}$$

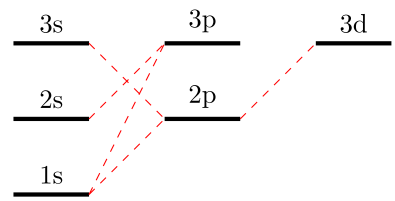
   Setting $I = I_{sat}$:

   $$\kappa(\omega, I) = \frac{N\sigma(\omega)}{2} = 500\frac{1}{\text{m}} \tag{10}$$

   and thus the absorption is equal to $1 - e^{-0.5} = 0.39$.

2. **Q2 Foot 7.6** *[3 pts]*

(a) Selection rule for electric dipole transitions: $\Delta l = \pm 1$



(b) The lifetime is given by $\tau = 1/\sum_i A_{21}^i$, where the sum runs over all the processes that lead to the final state.

$$\tau_{3s} = \frac{1}{6.3 \cdot 10^6} = 158.7\text{ns} \tag{11}$$

$$\tau_{3p} = \frac{1}{1.7 \cdot 10^8 + 2.2 \cdot 10^7} = 5.21\text{ns} \tag{12}$$

$$\tau_{3d} = \frac{1}{6.5 \cdot 10^7} = 15.38\text{ns} \tag{13}$$

The fraction of atoms that start in the 3p and end up in the 2s configuration is:

$$\frac{A_{21}^{2\text{s}\to 3\text{p}}}{A_{21}^{2\text{s}\to 3\text{p}} + A_{21}^{1\text{s}\to 3\text{p}}} = 0.115 \tag{14}$$

(c)

$$\tau = 1/\sum_i A_{21}^i, \text{ and } A_{21} \propto |D_{12}|^2 \propto |\langle 2|r|1\rangle|^2 \tag{15}$$

3p has two modes of decay, while 2p has only one, but if $A_{21}^{2\text{s}\to 3\text{p}} + A_{21}^{1\text{s}\to 3\text{p}} < A_{21}^{1\text{s}\to 2\text{p}}$, the lifetime for 2p configuration is still shorter. This is possible due to larger overlap of 2p with 1s wavefunctions in comparison to overlaps of the 3p with 2s and 3p with 1s wavefunctions. This is consistent with the fact that orbitals with higher main quantum number predominantly reside further from the nucleus.

(d) We can calculate the dipole matrix elements from the corresponding A_{21}:

$$|D_{21}|^2 = \frac{g_2}{g_1}\frac{3c^2}{\alpha}\frac{1}{\omega_{21}^3}A_{21} \tag{16}$$

$$\frac{g_2}{g_1} = \frac{2l_2+1}{2l_1+1}, \text{ and } \omega_{21} = 2\pi c R_\infty \left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right) \tag{17}$$

We thus arrive at:

$$|D_{12}^{1\text{s}\to 2\text{p}}| = 1.29 a_0 \tag{18}$$

$$|D_{12}^{1\text{s}\to 3\text{p}}| = 0.52 a_0 \tag{19}$$

$$|D_{12}^{2\text{s}\to 3\text{p}}| = 3.03 a_0 \tag{20}$$

$$|D_{12}^{2\text{p}\to 3\text{s}}| = 0.54 a_0 \tag{21}$$

$$|D_{12}^{2\text{p}\to 3\text{d}}| = 3.89 a_0 \tag{22}$$

(e) Saturation intensity is given by:

$$I_{sat} = \frac{\pi hc}{3\lambda^3 \tau} = \frac{\pi hc A_{21}}{3\lambda^3} \tag{23}$$

which we can calculate to be:

$$I_{sat}^{\text{2p}\to\text{3s}} = 4.64 \text{ W/m}^2 \tag{24}$$

$$I_{sat}^{\text{1s}\to\text{3p}} = 3.28 \cdot 10^4 \text{ W/m}^2 \tag{25}$$

(f) **Q3 Foot 7.9** *[1 pt]*
First, without taking into account the Doppler shift, population in the excited state is:

$$|c_2(t)|^2 = \left| \Omega \; \frac{\sin\big((\omega_0 - \omega)t/2\big)}{\omega_0 - \omega} \right|^2 \tag{26}$$

Now taking into account the Doppler shift ($\omega_0 - \omega \to \omega_0 - \omega + kv$) and Maxwell-Boltzmann distribution of velocities:

$$f(v) = \frac{1}{u\pi} e^{-v^2/u^2}, \; u = \sqrt{2k_b T/M} \tag{27}$$

we arrive at the expression:

$$|c_2(t)|^2 = \frac{e^2 \chi_{12}^2}{\hbar^2} |E(\omega)|^2 \int_{-\infty}^{\infty} dv \; \frac{\sin^2\big((\omega - \omega_0 + kv)t/2\big)}{(\omega - \omega_0 + kv)^2} f(v) \tag{28}$$

$$\begin{aligned} |c_2(t)|^2 = \Omega^2 \int_{-\infty}^{\infty} dv f(v) \frac{\sin^2\big((\omega - \omega_0 + kv)t/2\big)}{(\omega - \omega_0 + kv)^2} &= \\ = \frac{e^2 \chi_{12}^2}{\hbar^2} \Big| E(\omega) \Big|^2 \int_{-\infty}^{\infty} dv f(v) \frac{\sin^2\big((\omega - \omega_0 + kv)t/2\big)}{(\omega - \omega_0 + kv)^2} & \end{aligned} \tag{29}$$

Performing a change of variable to $x = (\omega_0 - \omega + kv)t/2$ and approximating the function $\text{sinc}^2(x)$ by $\delta(x)$:

$$|c_2(t)|^2 = \frac{e^2 \chi_{12}^2 |E(\omega)|^2 t}{2\hbar^2 k} f\Big(\frac{\omega - \omega_0}{k}\Big) \tag{30}$$

$$|c_2(t)|^2 = \frac{e^2 \chi_{12}^2 |E(\omega)|^2 tc}{2\hbar^2 u \omega_0 \sqrt{\pi}} e^{-c^2(\omega - \omega_0)^2/u^2\omega^2} \propto g_D(\omega) \tag{31}$$

as we set out to show.

3. **Q4 Foot 7.12** *[2.5+0.5 pts]*

(a)

$$H = H_0 + H' = H_0 + e\vec{E} \cdot \vec{r} \tag{32}$$

In the matrix form:

$$H_0 = \begin{bmatrix} E_2 - \frac{E_2 + E_2}{2} & 0 \\ 0 & E_1 - \frac{E_1 + E_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{E_2 - E_1}{2} & 0 \\ 0 & \frac{E_1 - E_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{\epsilon}{2} & 0 \\ 0 & -\frac{\epsilon}{2} \end{bmatrix} \tag{33}$$

and the perturbation matrix elements are:

$$H'_{11} = e\vec{E} \cdot \langle 1|\vec{r}|1\rangle = 0 \tag{34}$$

$$H'_{22} = e\vec{E} \cdot \langle 2|\vec{r}|2\rangle = 0 \tag{35}$$

$$H'_{12} = e\vec{E} \cdot \langle 1|\vec{r}|2\rangle = e\vec{E} \cdot \langle 2|\vec{r}|1\rangle = H'_{21} = V \tag{36}$$

Thus the total Hamiltonian:

$$H_0 = \begin{bmatrix} \frac{\epsilon}{2} & V \\ V & -\frac{\epsilon}{2} \end{bmatrix} \tag{37}$$

and the eigenvalues are:

$$E_{2/1} = \pm\sqrt{\epsilon^2/4 + V^2} \tag{38}$$

(b)

$$E_1 = \sqrt{\epsilon^2/4 + V^2} = -\frac{\epsilon}{2}\sqrt{1 + \frac{4V^2}{\epsilon^2}} \simeq -\frac{\epsilon}{2} - \frac{V^2}{\epsilon} \tag{39}$$

and thus the Stark shift for weak field is equal to:

$$\Delta E_1 = -\frac{V^2}{\epsilon} = -\frac{|\langle 1|H'|2\rangle|^2}{E_2 - E_1} \tag{40}$$

(c) "Sodium D Line Data" by Daniel A. Steck: the dipole matrix element for the transition 3s $\rightarrow$ 3p is equal to $3.52a_0$ and the energy of this transition is $\epsilon = 2.104$ eV. We can thus calculate the Stark shift:

$$\Delta E = -\frac{V^2}{\epsilon} = -\frac{(10^6 \text{ V/m} \cdot 3.52\ ea_0)^2}{2.104 \text{ eV}} = -1.66 \cdot 10^{-8} \text{ eV} \tag{41}$$

(d) In the first order perturbation theory the correction to the ground state is:

$$|1\rangle^{(1)} = eE\frac{\langle 2|z|1\rangle}{E_1 - E_2}\,|2\rangle^{(0)} = 8.9 \cdot 10^{-5}\,|2\rangle^{(0)} \tag{42}$$

Thus the percentage of excited state character mixed into the ground state is $100\% \cdot (8.9 \cdot 10^{-5})^2 = 7.9 \cdot 10^{-7}\%$.