PHYS 598 AQG HW6

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1. **Q1 "Light shifts and scattering rates"** *[3 pts]*

   (a)
   $$\frac{\Omega_{D2}}{\Omega_{D1}} = \frac{\langle J=1/2|er|J'=3/2\rangle}{\langle J=1/2|er|J'=1/2\rangle} = \sqrt{2} \tag{1}$$
   For large frequency detuning the AC Stark effect is:
   $$\Delta\omega_{D1} = \frac{\Omega_{D1}^2}{4\delta_{D1}}, \quad \Delta\omega_{D2} = \frac{\Omega_{D2}^2}{4\delta_{D2}} \tag{2}$$
   where $\delta_{D1} = \omega_L - \omega_{D1}$ and $\delta_{D2} = \omega_L - \omega_{D2}$. Thus the Stark shift vanishes when:
   $$\Delta\omega_{D2} + \Delta\omega_{D1} = 0 \;\Rightarrow\; \omega_L = \frac{2\omega_{D2} + \omega_{D2}}{3} = 379.5 \text{ THz} \tag{3}$$
   which corresponds to the wavelength $\lambda_L = 790$ nm.

   (b) The total rate of off-resonant scattering is given by:
   $$R = R_1 + R_2 = \frac{\Gamma}{2}\left(\frac{\Omega_{D1}^2/2}{\delta_{D1}^2 + \Omega_{D1}^2/2 + \Gamma^2/4} + \frac{\Omega_{D2}^2/2}{\delta_{D2}^2 + \Omega_{D2}^2/2 + \Gamma^2/4}\right) \tag{4}$$
   For a large frequency detuning $\delta \gg \Omega$, $\delta \gg \Gamma$:
   $$R \propto \left(\frac{\Omega_{D1}^2}{\delta_{D1}^2} + \frac{\Omega_{D2}^2}{\delta_{D2}^2}\right) \propto \left(\frac{1}{(\omega_L - \omega_{D1})^2} + \frac{2}{(\omega_L - \omega_{D2})^2}\right) \tag{5}$$
   Minimizing R with respect to ω_L:
   $$\frac{\partial R}{\partial \omega_L} = 0 \;\Rightarrow\; \omega_L = \frac{1}{1+2^{1/3}}(2^{1/3}\omega_{D1} + \omega_{D2}) \tag{6}$$
   which corresponds to the wavelength $\lambda_L = 788$ nm.

2. **Q2 Foot 10.4 "Evaporative cooling"** *[4 pts]*

   (a) The fraction of atoms lost:
   $$\frac{\Delta\mathcal{N}}{\mathcal{N}_{tot}} = \frac{A\int_\epsilon^\infty dE\; e^{-\beta E}}{A/\beta} = \int_{\beta\epsilon}^\infty d(\beta E)\; e^{-\beta E} = e^{-\beta\epsilon} \tag{7}$$

   (b) The mean energy per atom that remained:
   $$\frac{E_{left}}{\mathcal{N}_{left}} = \frac{A\int_0^\epsilon dE\; Ee^{-\beta E}}{A\int_0^\epsilon dE\; e^{-\beta E}} = \frac{1}{\beta}\left(1 - \frac{\beta\epsilon e^{-\beta\epsilon}}{1 - e^{-\beta\epsilon}}\right) \tag{8}$$
   and thus the fractional change in the mean energy per atom:
   $$\frac{\Delta\bar{E}}{\bar{E}} = \frac{\beta\epsilon e^{-\beta\epsilon}}{1 - e^{-\beta\epsilon}} \tag{9}$$

(c) For $\beta\epsilon = 3$:

$$\frac{\Delta\mathcal{N}}{\mathcal{N}_{tot}} = 0.05 \tag{10}$$

$$\frac{\Delta\bar{E}}{\bar{E}} = 0.16 \tag{11}$$

For $\beta\epsilon = 6$:

$$\frac{\Delta\mathcal{N}}{\mathcal{N}_{tot}} = 0.0025 \tag{12}$$

$$\frac{\Delta\bar{E}}{\bar{E}} = 0.0149 \tag{13}$$

Thus the ratio of fractional change of the mean energy to fraction change of the number of atoms are respectively 3.2 and 6.0, which clearly means that making more small evaporative cooling steps is a more efficient way to cool in comparison to larger evaporative steps. That is to say, evaporative cooling with a larger value of $\beta\epsilon$ is more efficient.

(d) In a harmonic potential:

$$E \propto r^2 \quad \text{or} \quad r \propto \sqrt{E} \;\Rightarrow n = \frac{\mathcal{N}}{V} \propto \mathcal{N}E^{-3/2} \tag{14}$$

Then taking into account $E \propto v^2$:

$$R_{col} \propto nv \propto \mathcal{N}E^{-3/2}E^{1/2} = \frac{\mathcal{N}}{E} \tag{15}$$

The change in R_{col} as we proceed with evaporation:

$$\Delta R_{col} \propto \frac{\partial R_{col}}{\partial \mathcal{N}}\Delta\mathcal{N} + \frac{\partial R_{col}}{\partial E}\Delta E = \frac{\mathcal{N}}{E}\left(\frac{\Delta\mathcal{N}}{\mathcal{N}} - \frac{\Delta E}{E}\right) \tag{16}$$

Since in the process of evaporative cooling $\left|\frac{\Delta\mathcal{N}}{\mathcal{N}}\right| < \left|\frac{\Delta E}{E}\right|$ and the mean energy decreases ($\frac{\Delta E}{E} < 0$), we conclude that the collision rate becomes larger.

3. **Q3 Foot 10.5 "The properties at the phase transition"** *[2 pts]*

In a harmonic potential Bose condensation happens approximately at:

$$T_C \simeq \frac{\hbar\bar{\omega}\mathcal{N}^{1/3}}{k_B} \tag{17}$$

Using the values $\omega_z = 2\pi \times 16$ Hz and $\omega_r = 2\pi \times 250$ Hz we calculate:

$$T_C \simeq \frac{\hbar(\omega_r^2\omega_z)^{1/3}\mathcal{N}^{1/3}}{k_B} = 478 \text{ nK} \tag{18}$$

The density at the transition can be estimated by:

$$n = \frac{2.6}{\lambda_{dB}^3} \tag{19}$$

where $\lambda_{dB} = \frac{h}{\sqrt{2\pi M k_B T}}$. Evaluating the density for rubidium atoms using T_C: $n = 1.3 \cdot 10^{14}$ cm^{-3}.

4. **Q4 Foot 10.10 "Derivation of the speed of sound"** *[5 pts]*

(a)

$$\psi = \psi_0 e^{-i\mu t/\hbar} + \delta\psi(t) = \Big[\psi_0 + ue^{i(kx-\omega t)} + v^* e^{-i(kx-\omega t)}\Big] e^{-i\mu t/\hbar} \tag{20}$$

$$i\hbar\partial_t\psi = \Big[\mu\psi_0 + \hbar\omega u e^{i(kx-\omega t)} + \mu u e^{i(kx-\omega t)} - \hbar\omega v^* e^{-i(kx-\omega t)} + \mu v^* e^{-i(kx-\omega t)}\Big] e^{-i\mu t/\hbar} \tag{21}$$

$$-\frac{\hbar^2}{2M}\nabla^2\psi = \frac{\hbar^2}{2M}\Big[uk^2 e^{i(kx-\omega t)} - v^* k^2 e^{-i(kx-\omega t)}\Big] e^{-i\mu t/\hbar} \tag{22}$$

$$|\psi|^2\psi \simeq \Big[\psi_0^3 + 2\psi_0^2\Big(ue^{i(kx-\omega t)} + v^* e^{-i(kx-\omega t)}\Big) + \psi_0^2\Big(u^* e^{-i(kx-\omega t)} + ve^{i(kx-\omega t)}\Big)\Big] e^{-i\mu t/\hbar} \tag{23}$$

where we kept only the first order terms in v and u. Assembling all these terms into Schrodinger equation and neglecting all the terms with v and u results in:

$$\mu = g|\psi_0|^2 \tag{24}$$

If we keep the first order terms in v and u, use Eq.24 and make a substitution $\delta\psi = \Big(ue^{i(kx-\omega t)} + v^* e^{i(kx-\omega t)}\Big)e^{-i\mu t/\hbar}$:

$$i\hbar\partial_t\delta\psi = \frac{\hbar^2 k^2}{2M}\delta\psi + 2g|\psi_0|^2\delta\psi + g\psi_0^2\delta\psi^* \tag{25}$$

(b) Writing out Eq.25 in terms of u and v:

$$\Big[\mu u + \hbar\omega u\Big]e^{i(kx-\omega t)} + \Big[\mu v^* - \hbar\omega v^*\Big]e^{-i(kx-\omega t)} = \Big[\frac{\hbar^2k^2}{2M}u + 2g|\psi_0|^2 u + g\psi_0^2 v\Big]e^{i(kx-\omega t)} + \Big[\frac{\hbar^2k^2}{2M}v^* + 2g|\psi_0|^2 v^* + g\psi_0^2 u^*\Big]e^{-i(kx-\omega t)} \tag{26}$$

Equating terms with the same time dependence:

$$\begin{bmatrix} \epsilon_k - \mu + 2g|\psi_0|^2 & g\psi_0^2 \\ g(\psi_0^*)^2 & \epsilon_k - \mu + 2g|\psi_0|^2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \hbar\omega \begin{bmatrix} u \\ -v \end{bmatrix} \tag{27}$$

where we introduced $\epsilon_k = \frac{\hbar^2 k^2}{2M}$.

(c) From Eq.27 using $\mu = g|\psi_0|^2$ we obtain:

$$\begin{bmatrix} \epsilon_k - g|\psi_0|^2 + 2g|\psi_0|^2 - \hbar\omega & g\psi_0^2 \\ g(\psi_0^*)^2 & \epsilon_k - g|\psi_0|^2 + 2g|\psi_0|^2 + \hbar\omega \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0 \tag{28}$$

$$\begin{bmatrix} \epsilon_k + g|\psi_0|^2 - \hbar\omega & g\psi_0^2 \\ g(\psi_0^*)^2 & \epsilon_k + g|\psi_0|^2 + \hbar\omega \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0 \tag{29}$$

$$\det\begin{bmatrix} \epsilon_k + g|\psi_0|^2 - \hbar\omega & g\psi_0^2 \\ g(\psi_0^*)^2 & \epsilon_k + g|\psi_0|^2 + \hbar\omega \end{bmatrix} = 0 \tag{30}$$

$$\omega = \frac{1}{\hbar}\sqrt{\epsilon_k^2 + 2\epsilon_k\mu} \tag{31}$$

$$\epsilon_k^2 + 2\mu\epsilon_k - (\hbar\omega)^2 = 0 \tag{32}$$

For small ϵ_k:

$$\omega \simeq \sqrt{2\epsilon_k\mu} = k\sqrt{\frac{\mu}{M}} \;\Rightarrow\; v_s = \frac{\omega}{k} = \sqrt{\frac{\mu}{M}} \tag{33}$$