Elastic Waves HW 2 SOLUTION

HW2.1 Given a finite anisotropic body of volume V with a traction free surface S, and a distribution of body forces in the interior **f**(**x**,t), such that the wave equation is

$$\rho(\vec{x})\, \ddot{u}_i - [c_{ijkl}(\vec{x})\, u_{k,l}]_{,j} = \vec{f}_i(\vec{x},t)$$

show that the solution (for any specified initial conditions at time zero) is unique.

This is a straightforward generalization of the proof given in class. Let there be two vector displacement fields **u**(**x**,t) and **w**(**x**,t) both satisfying the above PDE and the same Initial conditions and same traction boundary conditions. Subtract the two to find (where **z**(**x**,t) = **u**(**x**,t)-**w**(**x**,t) is the difference between them) that **z** satisfies the same PDE, but with no body force.

$$\rho(\vec{x})\, \ddot{z}_i - [c_{ijkl}(\vec{x})\, z_{k,l}]_{,j} = 0$$

We also recognize, because **u** and **w** satisfy the same initial conditions, that **z** obeys quiescent initial conditions $\vec{z}(\vec{x},t=0) = \dot{\vec{z}}(\vec{x},t=0) = 0$ We further note that , as **u** and **w** both satisfy the same surface traction boundary condition, the difference **z** satisfies traction free conditions.

Thus z satisfies the PDE without a force, quiescent ICs, and traction free surface conditions. Its energy density (which is positive definite)

$$\mathcal{E}_{\vec{z}}(\vec{x};t) = \tfrac{1}{2}\rho(\vec{x})\dot{z}_i\,\dot{z}_i \quad + \tfrac{1}{2}c_{ijkl}(\vec{x})z_{i,j}(\vec{x};t)z_{k,l}(\vec{x};t)$$

implies a total energy $E_z = \int \mathcal{E}_z\, dV$ that is i) zero at time zero due to the quiescent ICs and ii) is conserved due to the absence of flux $\hat{n}\cdot\vec{\mathcal{F}}_{\vec{z}} = 0$ across the surface (in turn due to z satisfying traction free conditions). Therefore $dE_z/dt = \int \mathcal{F}_z.\mathbf{n}\, dS = 0$. So total E_z remains equal to zero for all time, and **z** must be zero for all time and places. QED

One caveat: **z** = **constant** + **constant** * time is also a solution (with energy zero) , so the original PDE is actually unique only to within such.

HW 2.2 Show that $\sigma_{zz} = +\nu(\sigma_{xx} + \sigma_{yy})$ in plane strain (defined as a field of the form $\vec{u} = \hat{i}u_x(x,y;t) + \hat{j}u_y(x,y;t)$)

A **u** of that form implies a strain of the form

$$[\varepsilon] = \begin{bmatrix} u_{x,x} & (u_{x,y}+u_{y,x})/2 & 0 \\ (u_{x,y}+u_{y,x})/2 & u_{y,y} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding stress is

$$[\sigma] = \lambda\ Tr(\varepsilon)\,[\ I\ ] + 2\mu[\varepsilon] =$$

$$\lambda\,(u_{x,x}+u_{y,y})\,[\ I\ ] \ + \ 2\mu \begin{bmatrix} u_{x,x} & (u_{x,y}+u_{y,x})/2 & 0 \\ (u_{x,y}+u_{y,x})/2 & u_{y,y} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So

$$\sigma_{zz} = \lambda\,(u_{x,x}+u_{y,y}); \quad \sigma_{xx} = \lambda\,(u_{x,x}+u_{y,y}) + 2\mu u_{x,x}; \quad \sigma_{yy} = \lambda\,(u_{x,x}+u_{y,y}) + 2\mu u_{y,y}$$

From which one may construct

$$\sigma_{zz} = \lambda[\frac{\sigma_{xx}-\sigma_{zz}}{2\mu} + \frac{\sigma_{yy}-\sigma_{zz}}{2\mu}]$$

Solving for σ_{zz} gives

$$\sigma_{zz} = \frac{\lambda}{2\mu+2\lambda}[\sigma_{xx}+\sigma_{yy}] = \nu[\sigma_{xx}+\sigma_{yy}] \quad \text{QED}$$

HW2.3 Prove the continuity equation for the case of heterogeneous anisotropic medium with $c_{ijkl}(\mathbf{x})$ $\frac{\partial}{\partial t}\mathcal{E}(\vec{x};t) + \frac{\partial}{\partial x_i}\mathcal{F}_i(\vec{x};t) = 0$

We start with the definitions

$$\mathcal{E}(\vec{x};t) = \tfrac{1}{2}\rho(\vec{x})\dot{u}_i\,\dot{u}_i \quad + \tfrac{1}{2}c_{ijkl}(\vec{x})\varepsilon_{ij}(\vec{x};t)\varepsilon_{kl}(\vec{x};t)$$

$$\mathcal{F}_i(\vec{x};t) \;= -\dot{u}_j(\vec{x};t)\,c_{ijkl}(\vec{x})u_{k,l}(\vec{x};t)$$

and construct

$$\partial\mathcal{E}(\vec{x};t)/\partial t = \rho(\vec{x})\ddot{u}_i\,\dot{u}_i \quad + c_{ijkl}(\vec{x})\dot{\varepsilon}_{ij}(\vec{x};t)\varepsilon_{kl}(\vec{x};t)$$

$$\partial\mathcal{F}_i(\vec{x};t)/\partial x_i \;= -\dot{u}_{j,\,i}(\vec{x};t)\,c_{ijkl}(\vec{x})u_{k,l}(\vec{x};t) - \dot{u}_j(\vec{x};t)\{c_{ijkl}(\vec{x})u_{k,l}(\vec{x};t)\},_i$$

We substitute (in the first term of the first equation) for $\rho(\vec{x})\;\ddot{u}_i \;= [c_{ijkl}(\vec{x})\;u_{k,\,l}],_j$ to get

$$\partial\mathcal{E}(\vec{x};t)/\partial t = [c_{ijkl}(\vec{x})\;u_{k,\,l}],_j\,\dot{u}_i \quad + c_{ijkl}(\vec{x})\dot{\varepsilon}_{ij}(\vec{x};t)\varepsilon_{kl}(\vec{x};t)$$

So that

$$\frac{\partial}{\partial t}\mathcal{E}(\vec{x};t) + \frac{\partial}{\partial x_i}\mathcal{F}_i(\vec{x};t) =$$

$$[c_{ijkl}(\vec{x})\;u_{k,\,l}],_j\,\dot{u}_i \quad + c_{ijkl}(\vec{x})\dot{\varepsilon}_{ij}(\vec{x};t)\varepsilon_{kl}(\vec{x};t) - \dot{u}_{j,\,i}(\vec{x};t)\,c_{ijkl}(\vec{x})u_{k,l}(\vec{x};t) - \dot{u}_j(\vec{x};t)\{c_{ijkl}(\vec{x})u_{k,l}(\vec{x};t)\},_i$$

The first term cancels the last term (though you have to interchange i and j to see that, and use the symmetry of [[c]] in its first two indices.) Similarly, the second term cancels the third. (again using [[c]]'s symmetries) So $\partial_t\mathcal{E}(\vec{x};t) + \partial_i\mathcal{F}_i(\vec{x};t) = 0$ QED.

If we had had a body force source in the PDE $\rho(\vec{x})\;\ddot{u}_i \;= [c_{ijkl}(\vec{x})\;u_{k,\,l}],_j + f_i$ we'd find that the continuity eqn had a source term representing volume density of mechanical power input. $\partial_t\mathcal{E}(\vec{x};t) + \partial_i\mathcal{F}_i(\vec{x};t) = f_i\,\dot{u}_i$

HW2.4 A delta-function pulse $u_{incident} = \delta(t-x/c)$ may be Fourier decomposed in the form:

$$u(x,t) = (1/2\pi)\int \exp\{i\omega t - i\omega x/c\}d\omega.$$

where the integral runs from $-\infty$ to ∞.

The pulse is incident from the left in an infinite string (which has lineal density ρ per length and tension T) upon a mass-spring combination as illustrated. For each Fourier component (i.e for each ω) find the transmitted wave at x > 0

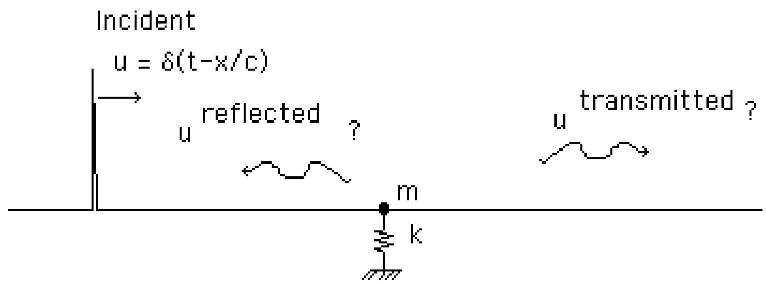
$Q(\omega)\exp\{i\omega t - i\omega x/c\}$

and find the wave at x < 0 in the form of the sum of incident and the reflected

$\exp\{i\omega t - i\omega x/c\} + R(\omega)\exp\{i\omega t + i\omega x/c\}$.

Then construct the transmitted wave at x > 0 $u_{transmitted}$ (= some function of x-ct) in the time domain by doing the Fourier integral re-composition. (It may be evaluated by Cauchy residue theorem.) You may assume the spring is stiff enough (large enough k) that the system is *underdamped*; thus the two poles of the integrand are located at points $\omega = \pm\omega_d + i\eta$. with non zero ω_d and positive η.

The spring k resists motion and transmits a force equal to k times the displacement of the mass. You may wish to note that the string is continuous at the mass, but its slope is not. You will need to construct the "jump" condition relating the slope at x = 0- to the slope at x = 0+.



Be careful – The integrand at large omega goes like 1/omega, which is too slow for the integral to converge. But if you take a time anti-derivative, the large omega behavior becomes like 1/omega^2, and the integral exists. Then after you evaluate the integral, you can take a time derivative.

The field on the right is $u = Q\exp(i\omega t - i\omega x/c)$.

It has slope at x = 0+ equal to $u'(x=0+)\ -i\omega Q/c\ \exp(i\omega t)$

The field on the left is $u = \exp(i\omega t)[\exp(-i\omega x/c) + R\exp(i\omega x/c)]$.

It has slope at (x = 0-) equal to $u'(x=0-) = -i\omega/c\ \exp(i\omega t)[1-R]$

Continuity at x = 0 demands

$$1 + R = Q$$

The jump condition at x = 0 tells us to set the net upward force from the string (the jump in the value of Tension times slope) onto the mass equal to the mass times its acceleration plus k times is displacement. (and its displacement is $Q\exp(i\omega t)$ and its acceleration is $-\omega^2 Q\exp(i\omega t)$)

$$(-m\omega^2 + k\}Q = T(i\omega/c)(1-R) - T(i\omega/c)Q$$

We substitute R = Q-1 and solve for Q:

$$Q = \frac{2Ti\omega/c}{-m\omega^2 + 2i\omega T/c + k}$$

(small sanity check: we confirm that Q = 1 ,i.e the transmitted wave is identical to the incident wave, when m = k = 0)

We now reconstruct the field in the time-domain by Fourier re-composition

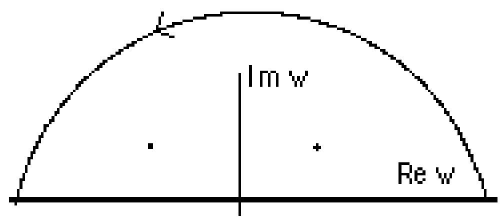
$$u(x>0,t) = \frac{1}{2\pi}\int d\omega \frac{2Ti\omega/c}{-m\omega^2 + 2i\omega T/c + k}\exp(i\omega(t - x/c))$$

The integral has some convergence difficulties, that are most simply removable by asking, not for u, but for its indefinite time integral

$$U(x,t) \equiv \int_{-\infty}^{t} u(x>0,\tau)d\tau = \frac{T}{\pi c}\int d\omega \frac{\exp(i\omega(t-x/c))}{-m\omega^2 + 2i\omega T/c + k}$$

This integral is convergent. It may be evaluated by using the Cauchy residue theorem. We need not introduce the infinitesimal positive imaginary shift $\omega+i\varepsilon$ because the integration does not go through a pole. But we do need to identify the poles of the integrand. They are at the roots of $-m\omega^2 + 2i\omega T/c + k$. This is the equation for a damped harmonic oscillator with stiffness k and mass m and damping 2T/c = 2ρc. One may interpret this as saying that the mass can oscillate freely in the absence of an incoming wave like a mass spring system, but with damping related to the energy radiated away along the string. The roots are (recall your damped oscillator formulas) at $\omega_\pm \equiv \pm\omega_d + i\eta$ with $\omega_d = \omega_n \{1-\zeta^2\}^{1/2}$ and $\eta = \zeta\omega_n$ where $\omega_n = \{k/m\}^{1/2}$ is called the undamped natural frequency and $\zeta = (T/c)/\{km\}^{1/2}$ is a dimensionless measure of the damping.

The complex ω plane looks like this, with the contour shown as extended into the upper half plane (valid if t-x/c > 0)

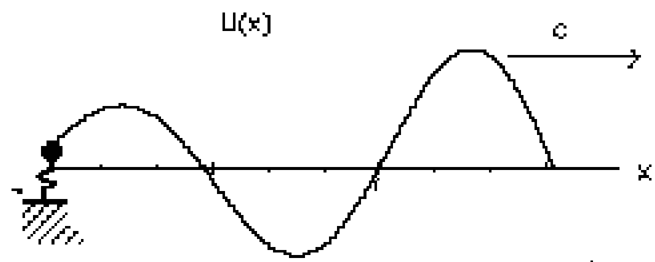


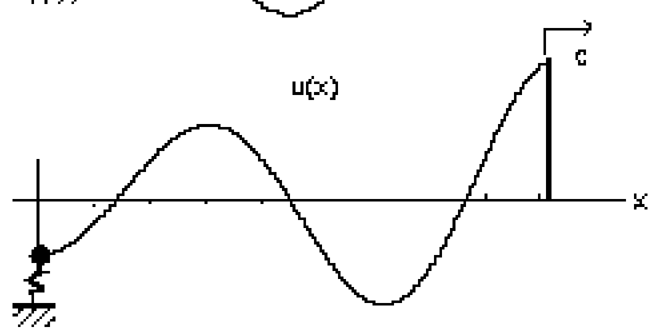
If alternatively t < x/c, then we should add a big semicircle in the lower half plane to the integration path (this adds zero) We then invoke the residue theorem, pick up no poles, and conclude that U vanishes for x > ct, as it ought, there can be no wave that arrives sooner than ct.

If t > x/c, we add a big semicircle in the upper half plane (as pictured) and pick up residues from two poles. The result is

$$U(x,t) = \frac{T}{\pi c}\sum_{\pm} 2i\pi \frac{\exp(\pm i\omega_d(t-x/c) - \eta(t-x/c))}{-2m(\pm\omega_d + i\eta) + 2iT/c}$$

$$= \frac{T}{c}\sum_{\pm} \frac{\exp(\pm i\omega_d(t-x/c) - \eta(t-x/c))}{\pm im\omega_d - m\eta + T/c}$$





The remaining algebra is a bit tedious. It is worth noticing that it is the sum of a quantity and its complex conjugate, so it is real, as it ought be. We can also see that the above describes an exponentially damped oscillation. Worked out in detail, (it helps to recognize that $m\eta = T/c$) it becomes

$$U = \frac{2T}{c}\exp(-\eta(t-x/c))\frac{\sin\omega_d(t-x/c)}{m\omega_d}$$

A snapshot of U(x) and u(x) =dU/dt = -cdU/dx at some positive time instant is given in the plot (your details may vary, depending on the numerical values of your parameters.) The illustrated waveforms propagate to the right without distortion at speed c.