Lecture 27 Diffuse Fields continued Monday Dec 7

Sample problem: An impulse acts normally on the surface of a homogeneous solid of finite volume.

1) How much work is done ?

Answer; we calculated that a few weeks ago p136. At short times, the problem looks like a half space. The result is given in terms of an integral over wave number q.

$$W\Delta\omega = -\frac{\omega}{2}\Delta\omega \,\mathrm{Im}\,\tilde{G}^{PL}_{zz}(x=y=z=0;\omega), \qquad with\,\tilde{G} = -\frac{1}{\mu}\frac{1}{2\pi}\frac{\omega^2}{c_T^2}\int_0^\infty q\,dq\,\frac{i\alpha}{R(q,\omega)}$$

The participation P of normal motion at the surface is related to W

$$W_i(x) = \sum_{n\in Band} |u_i^{(n)}(x)|^2/2 = N'(\omega)\Delta\omega < u_i^{(n)}(x)^2 > /2 = (N'(\omega)\Delta\omega/2\rho V)\ P_i(x)$$

Calculations (i.e doing the above integral) show this P to be about 2/3.

2) What fraction of the initial energy is deposited in Rayleigh waves?

Answer; we calculated that a few weeks ago; it came out to about 60% as I recall. The Rayleigh wave energy was due to <u>one</u> point of the integral – the rayleigh pole where R = 0.

3) What fraction of the energy is in Rayleigh waves after the field has scattered several times?

This is a classic diffuse field calculation. Scattering at corners and flaws causes energy originally in Rayleigh waves to mode convert to bulk P and S waves. (also bulk waves convert to R waves when they hit corners.) The final steady state equilibrium fraction would be *very* hard to calculate by considering mode-conversions at the corners It is much easier to calculate the ratio of the number of Rayleigh modes to the total number of modes. The number of Rayleigh modes $N_R(\omega)$, on an LxL square surface below some frequency ω, is the area of a circle of radius R= q_R L/2π (not the volume of a sphere as previously.) $N_R = \pi\,[L^2\, q_R(\omega)^2/4\pi^2\,]$ The number of Rayleigh modes in a narrow band Δω is then

$$\Delta\omega\ \ N_R'(\omega) = \Delta\omega\ \ \pi\,\partial_\omega\,[A\,q_R(\omega)^2/4\pi^2\,] = \Delta\omega\ A\ \omega/2\pi c_R^2\,.$$

where $A = L^2$ is the surface area (of the entire object, not just the top surface)

The number of total modes is (from a few pages ago)

$$\Delta\omega\ N'(\omega) = \Delta\omega\ V\omega^2[1/c_L^3 + 2/c_S^3]/2\pi^2$$

For simplicity, ignore the smaller term, the ratio is then

$(\pi A c_S / 2V\omega)\,(c_S / c_R)^2$

This is smaller than unity by a large factor, of the order of L in units of wavelength. The diffuse field assertion that each mode gets the same average enrgy then tells us that the energies are in the same ratio as the N. Most of the energy –at least if the object is large enough - is therefore in bulk shear waves. (On the other hand, because we measure at the surface, most of what we *detect* is Rayleigh waves.)

2) What is the mean square displacement in 'i' direction at generic points in the interior? From the definition of participation it is

$$P_i(\text{interior})\ W\Delta\omega / \rho V\omega^2$$

For a generic point in the interior we know P = 1/3, and we use the calculation of W from part (1)

3) What is the mean square displacement at generic points on the surface?

$$P_z(\text{surface})\ W\Delta\omega / \rho V\omega^2 \qquad \text{with } P \sim 2/3$$

4) What is the mean square normal displacement at the position of the source?
K = three times as much as part (3)

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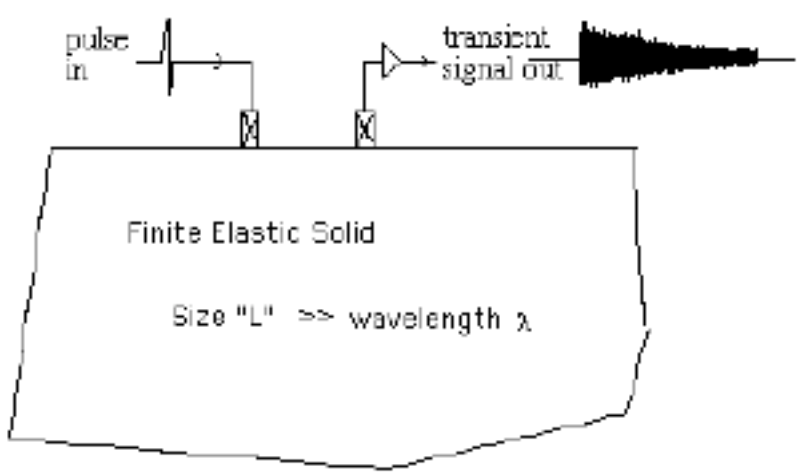
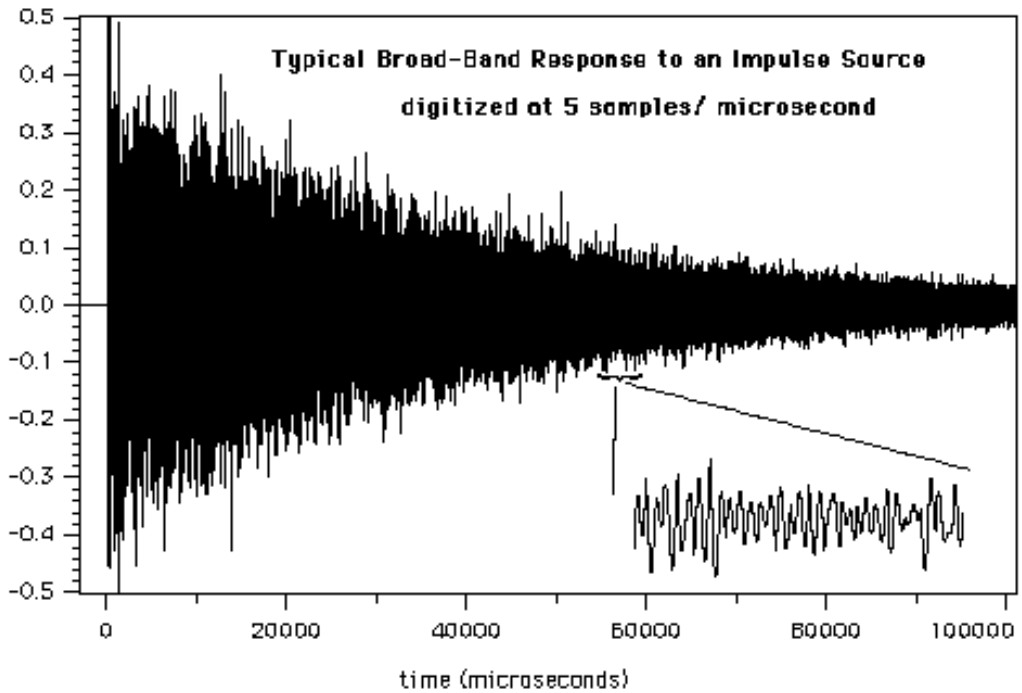
Back to the lab.
What do we expect to see by way of a signal at the receiver after this broadband pulse is applied to a source?

The figure shows what I typically measure





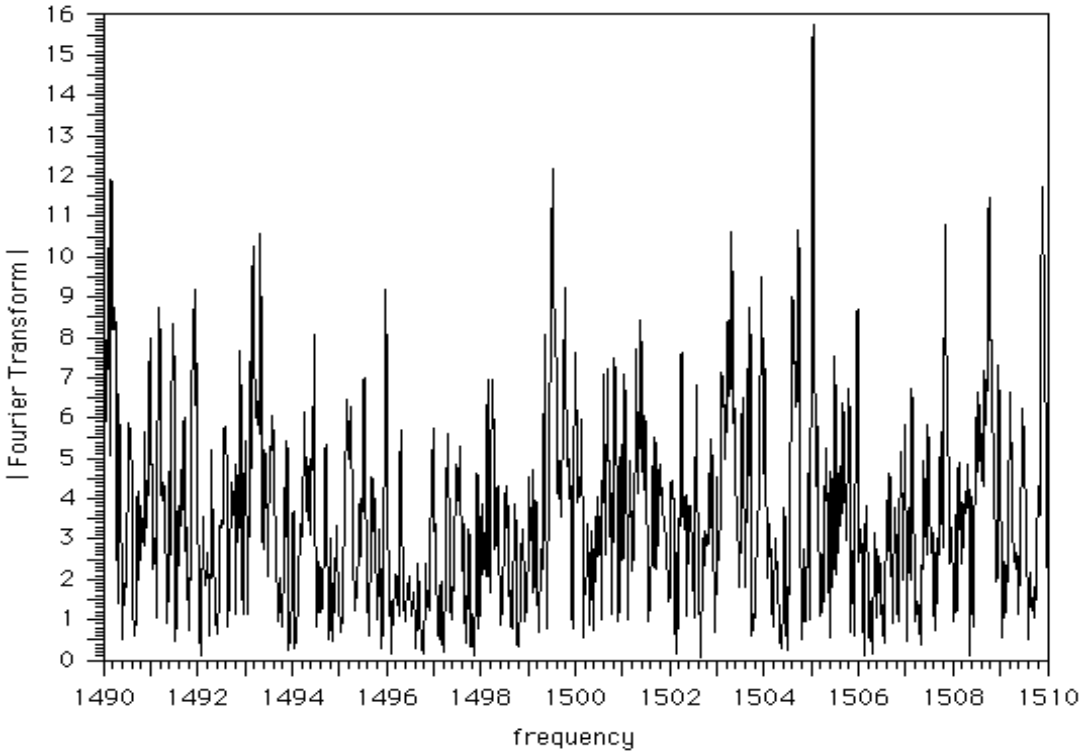
I see a noisy signal decaying with time. How do we analyze signals like this? What sort of signal processing should we apply to it? What characteristics of the source and receiver and specimen might we be able to extract from this signal?

The theory above said nothing about the decay – presumably there are dissipative processes that have not been included in the theory above that described the strength of the wave field and how the wave field is distributed in frequency and space. Dissipation can be included in the theory by an ad-hoc multiplication by $\exp(-\sigma t)$ where σ could be different at different frequencies.

The theory above *does* discuss mean square displacement – in various frequency bands $\Delta\omega$ at various positions. Inasmuch as σ should be expected to be frequency dependent, it makes sense then to band-pass filter the signal into various frequency bands. In order to capture the time-dependence of this envelope we should also window it into various time–windows.

The easy way to do this is to first time-window it (using time windows of short enough duration Δt. It is important to choose time windows of duration less than any time scales in the evolution of the signal envelope. For the above signal, the time scale of the envelope decay is 10's of miliseconds). Then within each time-window, we take a temporal FT and square it. If we'd used a *Discrete Fourier Transform* (FFT is a fast way to calculate it) we'd get the FT at a discrete set of frequencies f that are integer multiples of $\delta f = 1/\Delta t$. (up to a maximum frequency of $f_{Nyquist} = 2/\delta t$ where $1/\delta t$ is the digitization rate $\delta t << \Delta t$…. but that needn't concern us here.) Each point of the FT corresponds to a band of width $\Delta f = 1/\Delta t$ centered on some central frequency, a multiple of $1/\Delta t$. The result is then, for each time window of interest, an array of cokmplex FT veruss

frequency. A span of a plot might look like this… (from an actual measurement) of |FFT| versus f in KHz.



The points of this plot are spaced by δf, equal to about 0.1 kHz in this plot. Therefore Δt must have been 10 msec.

What is striking about this is that it has rapid noise-like fluctuations. Perhaps not surprising; it is an FFT of something that itself looked like noise. It is not hard to show that a real random Gaussian process, leads to fluctuations in $|FFT|^2$ that are of order 100%. No one point of an $|FFT|^2$ is a good representation of the mean. To get an estimate of the mean, one must do some averaging, it is simplest to sum over a band of frequencies $\Delta f = \Delta\omega/2\pi$, ie, a number n of the separate points of the FFT $n = \Delta f\, \Delta t$. Sufficient averaging to get a reliable number requires that this number $n = \Delta f\, \Delta t$ be large. We can show that, as long as the process is Gaussian, the result, after summing over the n points, gives us an estimate for the band-limited mean square signal at the time of the time-window, with a fractional error of one part in $\sqrt{n}$.

Additional averaging can sometimes be accomplished by varying the detection point and/or the source point. These can further improve the laboratory estimate of mean square signal. (an interesting question: how far does one have to move a source or receiver to get a fresh uncorrelated sample? Answer: typically half a wavelength suffices) Regardless, we'd like $\Delta f\, \Delta t$ to be large so as to minimize the fluctuations. This may well limit the time and frequency resolution of our measurements of mean square signal. If the envelope evolves rapidly (so that we need small Δt) , and if closely neighboring frequencies have very different behaviors (so that we need small Δf) , then the ease and utility of this kind of analysis is lessened.

It is helpful perhaps to summarize the various time and frequency scales that we've had to introduce so far.

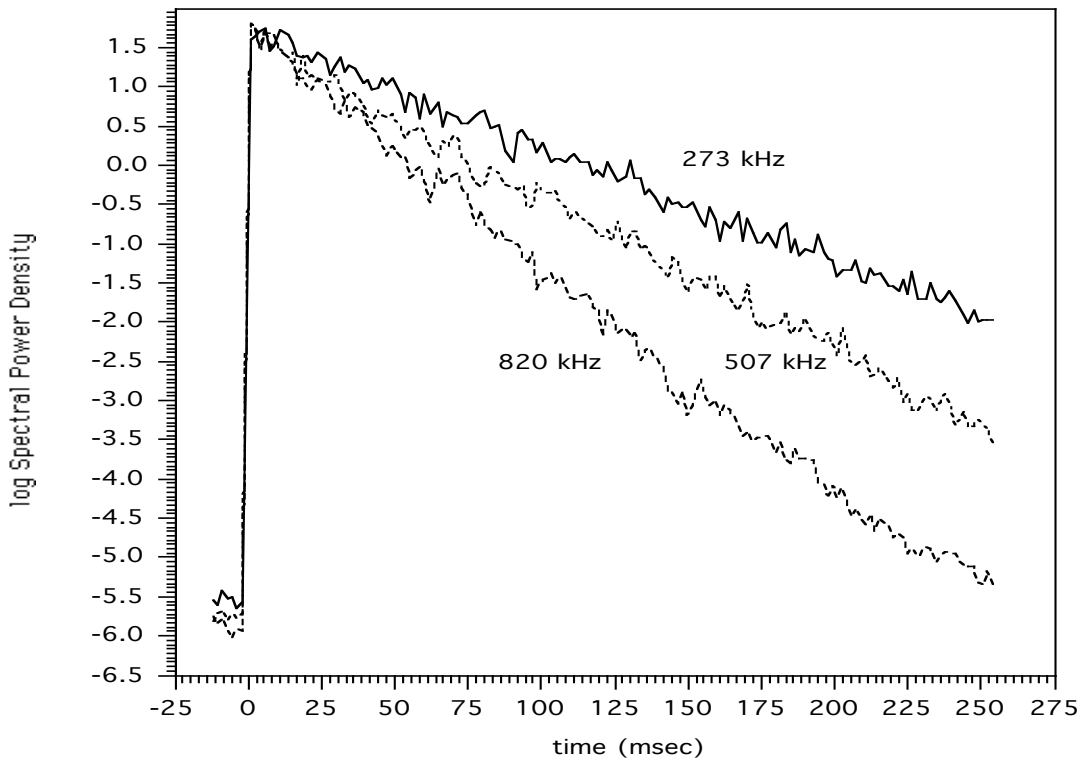
δt sampling time (usually of order μsec)

Δt = time window duration – your choice, but best to choose less that the time over which signals evolve.

δf= spacing in the discrete FT, imposed by the math to be 1/Δt

Δf = frequency bin width. Your choice. But best to choose smaller than the scale over which you expect signals to vary with frequency.

Here is the result of that analysis done on a signal like the above signal.



The (log) of the square of the FFT summed over three different frequency bands (central frequencies of 273 507 and 820 kHz) is seen to decay with time.

Just to be clear: we have plotted $\log(\sum_{band} | FFT_{window\Delta t} |^2)$. The sum over the band is over all points of the FFT within the bin of width Δf centered on the frequency that labels the curves. The FFT was of one time window of the raw signal, a window of duration Δt centered on the t indicated on the horizontal axis..

Note that the energy at higher frequencies decays more quickly. The Δt appear from the plot to be slightly less than 1 msec. The band widths Δf were, I am guessing about 80 kHz. The product was therefore probably 64 (for technical reasons it is often easiest to have it be a power of 2), so fluctuations should be a part in √64 or about 12%. Any

random quantity $p = < p > \pm \delta p$ has a $\log_e$ that is (from the Taylor series for log) $\log p \approx < \log p > \pm \delta p / < p >$, so each of these logs should have an error of 0.12. This is consistent with the observed fluctuations in the plot.

I also observe that the 820 kHz bin shows - at late time - that it approaches the background electronic noise level (indicated by the level at negative times.) The time range over which one can hope to get useful information is therefore limited by the duration over which the signal remains above noise. This can maybe be ameliorated by improving signal to noise ratios - by improving transducers or repetition averaging the measurement.

There are two system-dependent parameters that could be extracted from this data set. One could fit these to straight lines and recover

1) the decay rates – the frequency dependent rate at which energy dissipates. This could be a useful material property – although the decay rate may also be affected by losses into the air or to the transducers, also surface treatments. and

2) the intercepts at time zero – which correspond to the the initial energy deposition - as filtered by the receiving transducer and amplifier. This is often more a function of the transducers than of the material. However, in a thick plate we perhaps recall that the energy deposition has discontinuities - as a function of frequency- at the cutoffs of the various branches of Lamb wave propagation.(HW11.5) In this case, the intercepts give information on the geometry near the source and receiver. See below.

One could also confirm the predictions about the fluctuation levels. There are structures in which we find fluctuations that are much stronger than a part in $\sqrt{\Delta f \Delta t}$. This could be an indication that the signal was not a Gaussian random process.

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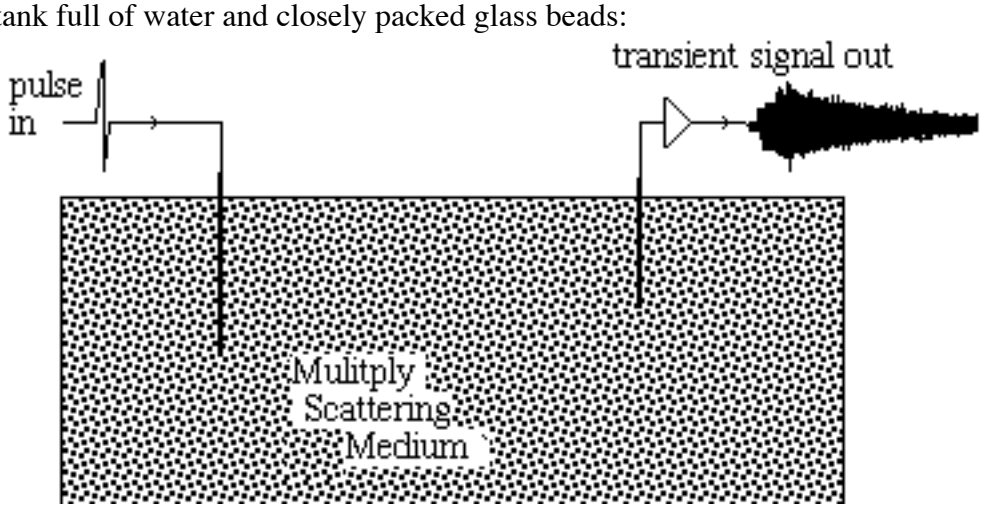
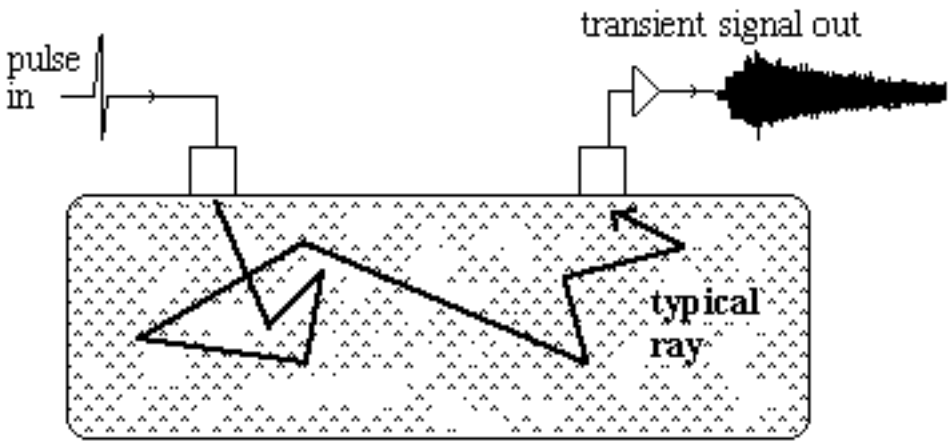
Another kind of system – one that is strongly multiply scattering.

A fish tank full of water and closely packed glass beads:



Or a chunk of concrete with lots of aggregate



In both of these systems, and if source and detector are far enough apart, direct signals from the source to the detector are heavily attenuated by the scattering. Nevertheless there is a detected signal that must correspond to waves that have scattered many times and gone by a complicated path.

Let us confine ourselves to times after the source acts that are sufficiently late that every ray can be expected to have scattered a few times, or more. I.e. t > l/c where l is the mean free path – the typical distance gone by a ray before it scatters. (this depends on the density and strength of the scatters) l is related to the attenuation of a ensemble average G by $l \sim 1/2\alpha$. In the fish tank, l was a few mm. In this regime we model the energy evolution by a diffusion equation (in turn justified by a picture of the rays as undergoing a random walk)

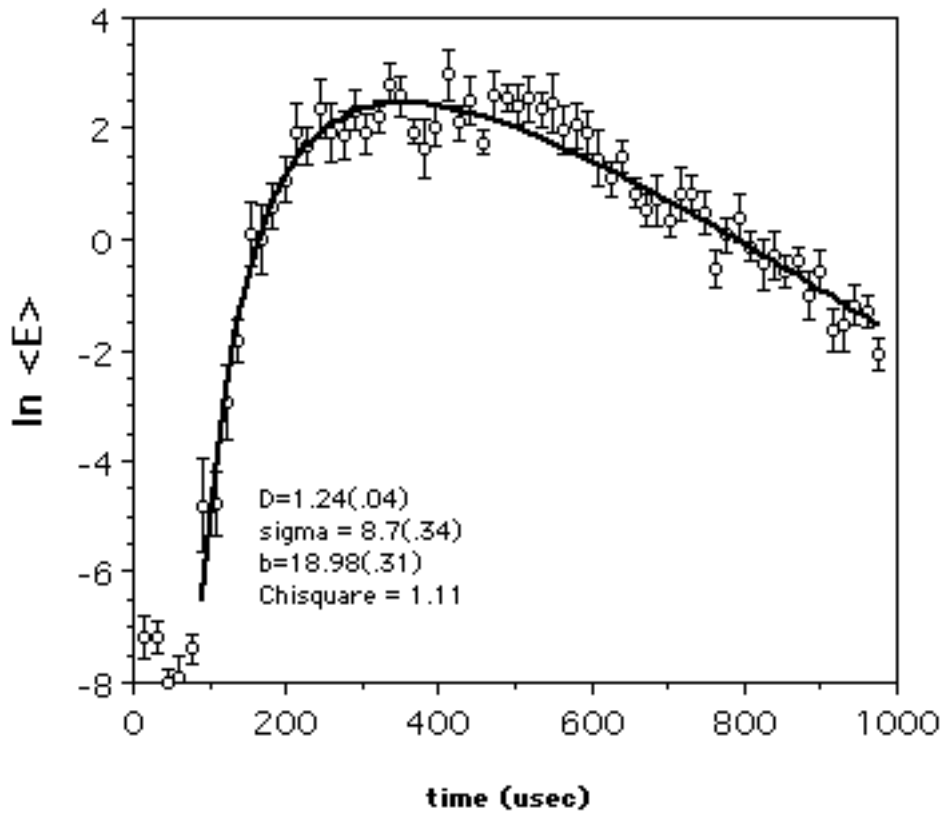
$$\partial_t E(\vec{x},t) = D\nabla^2 E(\vec{x},t) - \sigma E(\vec{x},t) + \delta^3(\vec{x}-\vec{y})\delta(t)$$

This describes a quantity (spectral energy density $E(\mathbf{x},t)$ per frequency per volume – while we recognize as proportional to our signal strength in volts after FTing and squaring and summing over a frequency bin.) that has its source at **y** at time zero. It is dissipated at a rate σ, and diffuses with diffusivity D. The random walk picture suggests that D can be identified with lc/3. Thus measurements of D give indications about the microstructure. This diffusion equation PDE is supplemented - in most circumstances in ultrasonics - with insulating boundary conditions $\hat{n}\cdot\nabla E(\vec{x},t)_{surface} = 0$ In the absence of complicating boundaries the solution is a simple spreading Gaussian envelope : (r being the distance | **x**-**y** |

$$E(r,t) = (4\pi Dt)^{-3/2}\ \exp(-r^2/4Dt)\ \exp(-\sigma t)$$

The solution takes a slightly different form in 1-d or 2-d, and a more complicated form if boundaries are significant.

Here is a result ($|\mathrm{FFT}|^2$ summed over a frequency band, versus time) from the fish tank for a band centered on 650 kHz (with a width Δf of about 100 kHz)



We see a plot of the log of the mean square signal . Time windows Δt wer apparentely chosen to be about 20 μsec. Frequency bandwidth was 100kHz, leading to expected $\delta f\ \Delta t = 2$ and fluctuations of a part in $\sqrt{2}$. The error bars are smaller than 0.71, however,, because there was some spatial averaging as well. The paper indictes that these energy densities are averaged over 6 or 7 realizations obtained by moving the transducers or restirring the slurry.

Also plotted is a best fit of the solution of a diffusion equation (solid curve), and we see that it fits very well. The fit process extracts best estimates for the dissipation σ and the Diffusivity D. The former is mostly due to viscous drag at the bead surfaces, the latter to scattering; thus both are microstructural parameters of potential interest in characterizing the composite medium.

These ideas (about how the diffuse field energy density evolves according to the Diffusion equation) are sometimes applied to concrete, with some success; the challenges as I understand them are 1) dissipation is so strong that one needs lots of spatial averaging and good signal to noise to get sufficient time range and time resolution to retrieve D with good accuracy. 2) The bodies can be statistically inhomogeneous, thus making D be a function of position and complicating the fits.

A few more words on when we can do a diffuse field analysis.

1) Is the signal sufficiently long lived above noise (this can be enhanced by more sensitive transducers and/or repetition averaging, and/or minimizing absorption or other loss mechanisms) that you can see the desired evolution ?

2) Is this long-lived signal due to rays that have randomly explored at least part of the volume?

3) Can you make $\Delta f\ \Delta t$ times the number of independent receiver and source positions large enough to get good accuracy? Δf is often limited by the expected frequency variation of the parameters you wish to extract. Δt is limited by the time scales over which the field evolves.

There are a couple more time scales worth keeping in mind.

Absorption time $1/\sigma$ is one of them. Closely related is duration over which signal remains above noise (depends also on signal to noise ratio as well as absorption time) .

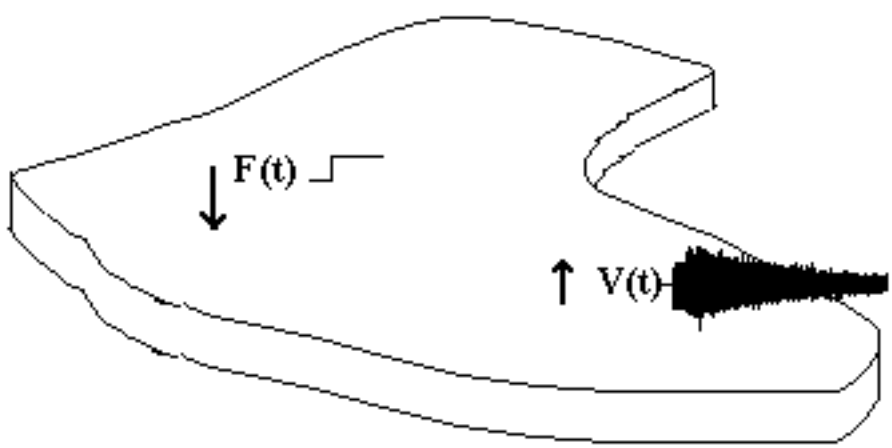
A more subtle time scale is the time for rays to mix throughout a volume. In homogeneous finite bodies it is of the order of a couple of transit times L/c. In a multiply scattering medium it is of order L^2/D. In a pair of weakly coupled subsystems (a solid block that is nearly cut in half for example) it is the expected time for a randomly moving ray to find it way from one side to the other.

An more subtle time scale is the so-called "Heisenberg time" T_H equal to the time needed to resolve the modes for a given volume. $T_H = (1/2\pi)\ N'(\omega) = dN/df$. It scales with the volume and with the square of the frequency and is in most applications longer than any other time scales of interest.

What makes it relevant is that - if the mixing time is longer than the Heisenberg time, then the system knows it has modes before the energy mixes through the volume. But the modes are standing waves so no further transport can take place.

It is also relevant as the minimum signal duration needed if you wish to identify the modes.

Here is one of my early experiments to test the predictions of the theory. I had a thick plate, with normal step force (a broken capillary) on the top surface. I had a normal displacement sensor at another position on the top surface. The received signal looked something like those illustrated above.

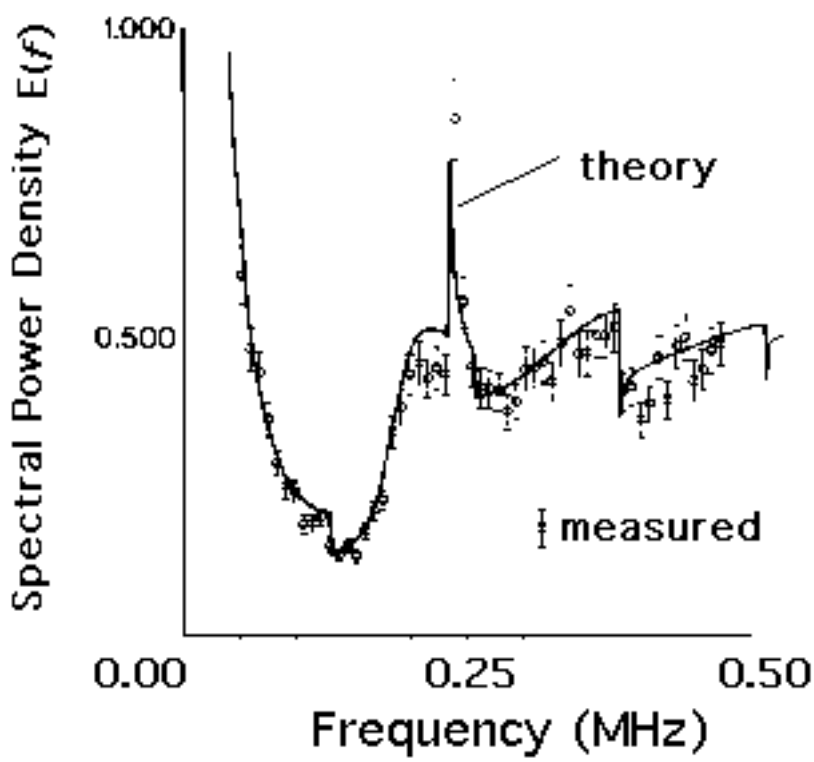


The frequency-and time-resolved signal^2 was evaluated as described above. Our theory for it is essentially as described above, but with the extra *ad hoc* factor of exp(-σt not inlcuded in basic diffuse field theory) Therefore we predict

$$< \psi(t;\omega)^2 >_{band} = 2\frac{W_z(\omega)W_z(\omega)}{\omega^2 N'(\omega)}\Delta\omega \ \ |\tilde{F}(\omega)|^2 |\tilde{R}(\omega)|^2 \ \exp(-\sigma t)$$

Where F and R are the source and receiver functions and σ is the decay rate. $\tilde{F}(\omega)$ is presumably the FT of a step function. R depends on the receiving transducer and the amplifier.

Fits to decaying exponentials gave -for each frequency of interest - the decay rate σ (which was of little interest to me) and the intercepts at time zero. These intercepts are, according to the above expression, proportional to two factors of the work W done by a normal impulse on the top surface, theoretically obtainable from the Im G for a thick plate. And one inverse factor of modal density N' of a thick plate. This follows from calculations of Lamb waves. The frequency dependence of this should be striking; W and N' both have discontinuities, and even singularities, at special frequencies corresponding to cutoffs of the Lamb and guided SH waves. The frequency dependence is also affected by the F and R, but their influences were eliminated by dividing by the same measurement done on a large irregular block for which the factors of W and N' are very smooth in frequency, with no influence from the finite thickness and no discontinuities. The result was….



which I took as a nice corroboration of the principles of diffuse field theory.