Lecture 22

- Final presentation topic Ideas on course website
    - Email me your choice by 11/11/2023
- Presentation logistics: 11/28, 11/30, 12/5 for presentations

  20 min + 5 mins for questions

- "New Horizons in Condensed Matter Physics" workshop 11/3 – 11/5

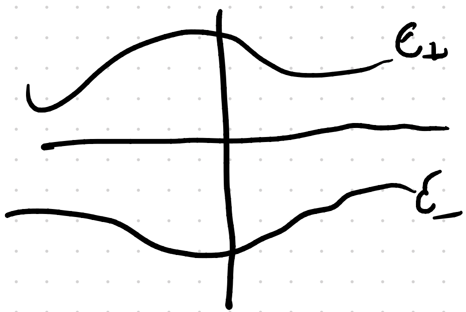
Last time: 2D system w/ inversion symmetry

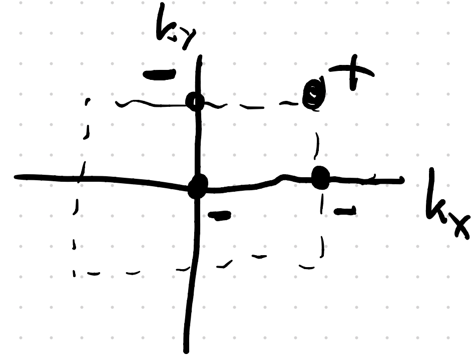
$$h(k) = (\Delta + \Delta\cos k_y + t_1 \cos k_x)\sigma_z + t_2 \sin k_x \sigma_y + t_2 \sin k_y \sigma_x$$

$\Delta, t_1, t_2 > 0$

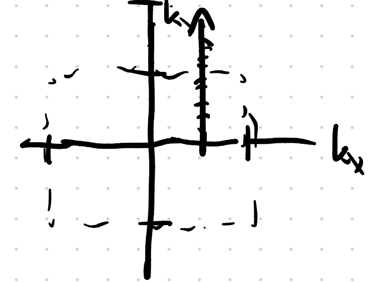
$\Delta > t_1/2$

$$\varepsilon_\pm = \pm\sqrt{(\Delta + \Delta\cos k_y + t_1\cos k_x)^2 + (t_2 \sin k_x)^2 + (t_2 \sin k_y)^2}$$





$$W^{-}_{\frac{2\pi}{a}\leftarrow 0}(k_x) = \hat{u}^{\dagger}_{-0k_x} \prod_{k_y}^{\frac{2\pi}{a}\leftarrow 0} P(k_x,k_y)\cdot \hat{u}_{-0k_x}$$

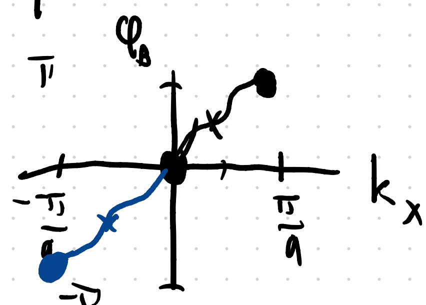


$$W^{-}_{\frac{2\pi}{a}\leftarrow 0}(0) = \det W_{\frac{2\pi}{a}\leftarrow 0}(k_x=0) = (-1)(-1) = +1$$

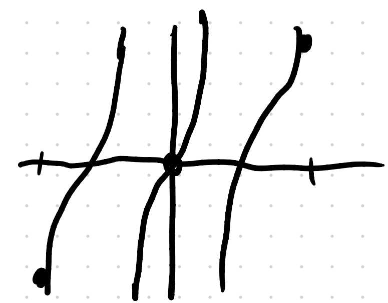
$$\rightarrow \operatorname{Im}\log\det W(k_x=0) \equiv \varphi_B(k_x=0) = 0$$

$$W^{-}_{\frac{2\pi}{a}\leftarrow 0}(k_x=\pi/a) = (+1)(-1) = -1$$

$$\rightarrow \varphi_B(k_x=\tfrac{\pi}{a}) = \pi$$



or



$$W(k_x) \qquad W(-k_x)$$

φ_B is related to eigenvalues of PxP

Inversion : $PxP \to -PxP$

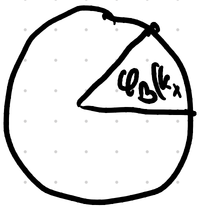
$$PxP|W_{0k_x}\rangle = \frac{1}{2\pi}\varphi_B(k_x)|W_{0k_x}\rangle$$

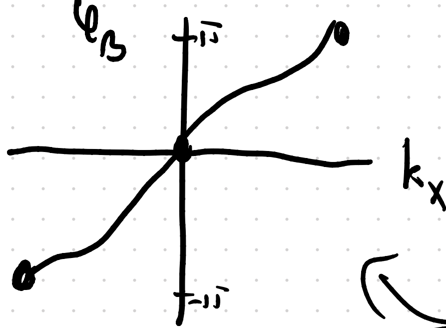
$$U_I\left[PxP|W_{0k_x}\rangle\right] = \frac{1}{2\pi}\varphi_B(k_x)U_I|W_{0k_x}\rangle$$

$$PxP\,U_I|W_{0k_x}\rangle = -\frac{1}{2\pi}\varphi_B(k_x)U_I|W_{0k_x}\rangle$$

$|W'_{0-k_x}\rangle \equiv U_I|W_{0k_x}\rangle$ has crystal momentum $k'_x = -k_x$

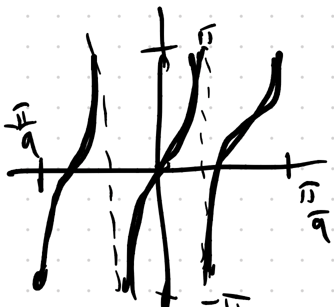
$$PxP|W'_{0-k_x}\rangle = -\frac{1}{2\pi}\varphi_B(k_x)|W'_{0-k_x}\rangle$$





$$C \equiv \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} \partial_x \varphi_B(k_x)\, dk_x$$

$$\boxed{C=1}$$



$C=3$

# of times the Berry phase winds from $-\pi$ to π

<u>Chern number</u> –

$$e^{i\varphi_B(k_x+\frac{2\pi}{a})} = e^{i\varphi_B(k_x)}$$

$$\varphi_B\left(k_x+\frac{2\pi}{a}\right) = \varphi_B(k_x) + 2\pi n$$

$n \in \mathbb{Z}$

$$\varphi_B(k_x) = nak_x + f(k_x)$$

$$f\left(k_x + \frac{2\pi}{a}\right) = f(k_x)$$

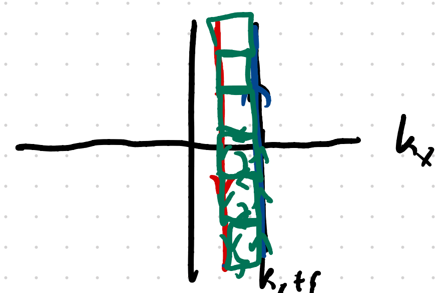
$$C = \frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_x \frac{\partial}{\partial k_x}\left(n a k_x + f(k_x)\right)$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dk_x\, n a + \cancel{f'(k_x)} \to 0$$

fundamental thm of calculus

$$= n$$

$$\frac{\partial}{\partial k_x}\varphi_b(k_x) = \frac{\partial}{\partial k_x}\,\mathrm{Im}\log\det W(k_x)$$



$$= \mathrm{Im} \lim_{\varepsilon \to 0} \frac{\log\det W(k_x+\varepsilon) - \log\det W(k_x)}{\varepsilon}$$

$$= \mathrm{Im} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \log\det\left[W(k_x+\varepsilon)\, W^{-1}(k_x)\right]$$

$$\frac{\partial}{\partial k_x}\varphi_B(k_x) = \int dk_y\, \mathrm{tr}\left[\Omega_{xy}(k_x,k_y)\right]$$

$$\Omega_{xy} = \frac{\partial}{\partial k_x}A_y - \frac{\partial}{\partial k_y}A_x - i[A_x, A_y]$$

$$\boxed{C = \frac{1}{2\pi}\oiint_{BZ} dk_x\, dk_y\, \mathrm{tr}\,\Omega_{xy}(k_x,k_y)}$$

① Physical meaning of C

Hybrid Wannier functions: $|W_{R,k_x+\frac{2\pi}{a}}\rangle = |W_{R+Ca\hat{y},k_x}\rangle$

$$P_y P\,|W_{R_y k_x+\frac{2\pi}{a}}\rangle = \underbrace{R_y + \frac{1}{2\pi}\varphi_B\left(k_x+\frac{2\pi}{a}\right)}_{(R_y+Ca)+\frac{1}{2\pi}\varphi_B(k_x)}\,|W_{R_y,k_x+\frac{2\pi}{a}}\rangle$$

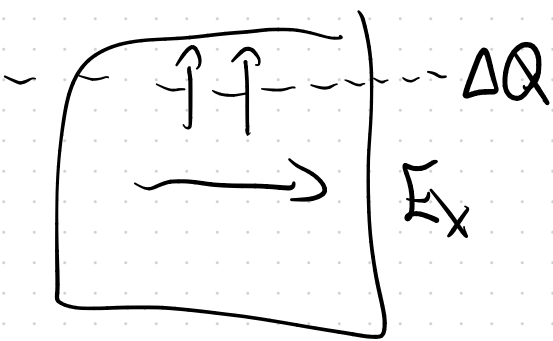
Electric field $\vec{E} = E_0\hat{x} = -\dfrac{\partial\vec{A}}{\partial t}$

$$\vec{A} = -E_0 t\,\hat{x}$$

$$\vec{p} \to \vec{p} - q\vec{A} = p + qE_0 t\,\hat{x}$$

$$|W_{R_y k_x}\rangle \to |W_{R_y k_x(t)}\rangle \qquad k_x(t) = k_x + qE_0 t$$

$$T = \frac{2\pi\hbar}{aE_0 q}$$



$$\Delta Q = C q N_x$$

$$I_y = \frac{\Delta Q}{T} = C\frac{q^2}{2\pi\hbar}V_x$$

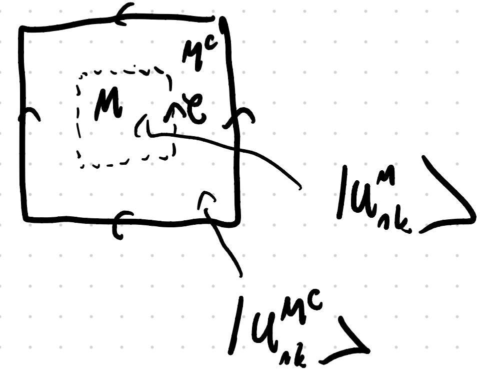
$$G_H = I_y/V_x = C\frac{q^2}{2\pi\hbar}$$

C is the quantum (anomalous) Hall conductance

C is a topological invariant - it canot change unless we close an energy ga

Claim: $C \neq 0 \rightarrow$ the occupied bands do not have exponentially localized Wannier functions

$$C = \frac{1}{2\pi} \oiint_{BZ} \operatorname{tr} \Omega \, d^2k$$



$$= \frac{1}{2\pi} \left[\oiint_{M} \operatorname{tr} \Omega \, d^2k + \oiint_{M^c} \operatorname{tr} \Omega \, d^2k \right]$$

M: $\operatorname{tr} \Omega = \operatorname{tr} \nabla \times A_M$

$$A_M = i \langle u^M_{nk} | \partial_k u^M_{nk} \rangle$$

$$= \frac{1}{2\pi} \left[\oint_{\mathcal{C}} A_M \cdot d\vec{\ell} - \oint_{\mathcal{C}} A_{M^c} \cdot d\vec{\ell} \right]$$

M^c: $\operatorname{tr} \Omega = \operatorname{tr} \nabla \times A_{M^c}$

$$= \frac{1}{2\pi} \oint_{\mathcal{C}} (A_M - A_{M^c}) \cdot d\vec{\ell}$$

on $\mathcal{C}$: $\operatorname{tr} \nabla \times A_{M^c} = \operatorname{tr} \nabla \times A_M$

$$\operatorname{tr} \vec{A}_{M^c} = \operatorname{tr} \vec{A}_{M^c} + \nabla \varphi$$

$$= \frac{1}{2\pi} \oint_{\mathcal{C}} \nabla_{\vec{k}} \varphi \cdot d\vec{l} \in \mathbb{Z}$$

$$|u^{gc}_{nk}\rangle = e^{i\varphi} |u^{M}_{nk}\rangle$$

$\exists$ some point where our gauge transformation stops being defined

$C \neq 0$ means I can't find a gauge where $|u_{nk}\rangle$ are analytic in k_x & k_y

$$\boxed{|\psi_{nk}\rangle} = \sum_{R} e^{ik\cdot R} |W_{nR}\rangle$$