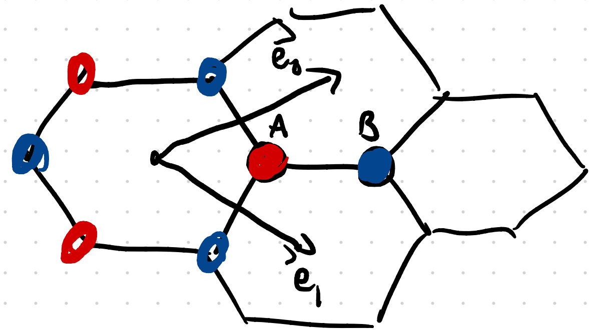
Lecture 24



Basis states $|\tau\sigma\vec{R}\rangle$

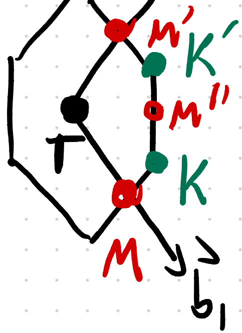
$$h(\vec{k}) = \begin{pmatrix} 0 & q(k) \\ q^*(k) & 0 \end{pmatrix} \otimes \sigma_0$$

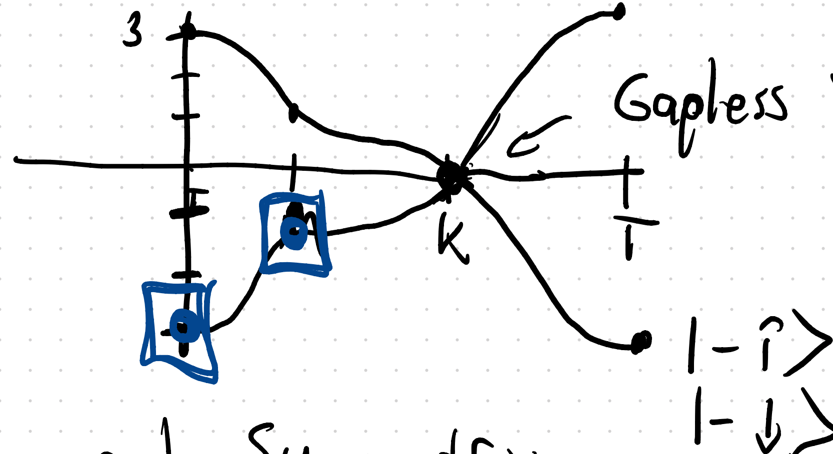
$$q(k) = t\left[e^{-\frac{2\pi i}{3}(k_1+k_2)} + e^{\frac{2\pi i}{3}(2k_1-k_2)} + e^{\frac{2\pi i}{3}(2k_2-k_1)} \right]$$

$$\varepsilon_\pm(\vec{k}) = \pm |q(\vec{k})|$$

b_2

Brillouin Zone





Inversion and Time-reversal Symmetry

$$B(I) = \tau_x \sigma_0$$

$$B^\dagger(I)\, h(k)\, B(I) = (\tau_x \sigma_0)\left[\begin{pmatrix} 0 & q(k) \\ q^*(k) & 0 \end{pmatrix}\sigma_0\right](\tau_x \sigma_0)$$

$$= \begin{pmatrix} 0 & q^*(k) \\ q(k) & 0 \end{pmatrix}\sigma_0$$

$$= \begin{pmatrix} 0 & q(-k) \\ q^{+}(-k) & 0 \end{pmatrix} \sigma_0 = h(-k)$$

Time-reversal symmetry

$$B(T) = i\tau_0 \sigma_y \mathcal{K}$$

$$B(T)\; h(k)\, B^{-1}(T) = (i\tau_0\sigma_y)\, h^*(k)\, (-i\tau_0\sigma_y)$$

$$= h^*(k) = h(-k)$$

Negative energy state inversion eigenvalues

$$\Gamma: \; h(\Gamma) = h(0) = \begin{pmatrix} 0 & q(0) \\ q^*(0) & 0 \end{pmatrix} \sigma_0$$

$$= 3t\, \tau_x \sigma_0$$

← negative energy eigenstates

negative eigenstates of $\tau_x \sigma_0 = B(I)$

→ negative energy states have $(-I, -I)$ inversion eigenvalue.

M: $\vec{k} = \frac{1}{2}\vec{b}_1$

$$B(I)\, h\left(\tfrac{1}{2}\vec{b}_1\right) B^\dagger(I) = h\left(-\tfrac{1}{2}\vec{b}_1\right)$$

$$= h\left(\tfrac{1}{2}\vec{b}_1 - \vec{b}_1\right)$$

$$= V(b_1)\, h\left(\tfrac{1}{2} b_1\right) V^\dagger(b_1)$$

$$V(b_1) = \begin{pmatrix} e^{i(\vec{b}_1 \cdot \vec{r}_A)} & 0 \\ 0 & e^{i(b_1 \cdot \vec{r}_B)} \end{pmatrix}$$

$$h\left(\tfrac{1}{2}b_1\right) = \left[V^\dagger(b_1)B(I)\right] h\left(\tfrac{1}{2}b_1\right)\left[V^\dagger(b_1)B(I)\right]^\dagger$$

at M: inversion is represented by

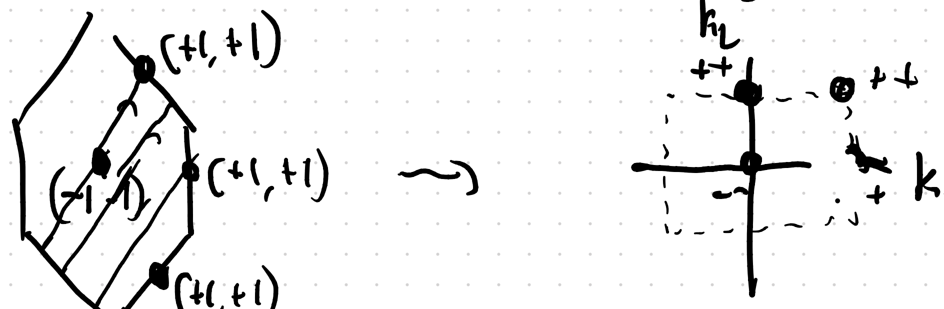
$$V^\dagger(b_1)B(I) = \begin{pmatrix} e^{-i(b_1\cdot r_A)} & 0 \\ 0 & e^{-i(b_1\cdot r_B)} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sigma_0$$

$$= \begin{pmatrix} 0 & e^{-i(b_1\cdot r_A)} \\ e^{-i(b_1\cdot r_B)} & 0 \end{pmatrix}\sigma_0$$

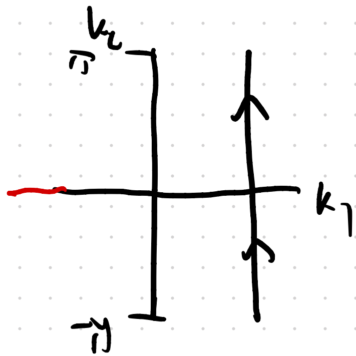
$$= \begin{pmatrix} 0 & e^{-2\pi i/3} \\ e^{2\pi i/3} & 0 \end{pmatrix}$$

$$= -h\left(\tfrac{1}{2}\vec{b}_1\right)/t$$

$\Rightarrow$ at M_i negative energy states have $(+1, +1)$ inversion eigenvalues



$$W^{\sigma\sigma'}_{2\pi\leftarrow 0}(k_1) = \langle u^{\sigma}_{-0} | V(\vec{b}_2) \prod_{k_2'}^{2\pi\leftarrow 0} P(k_1, k_2') | u^{\sigma'}_{-0} \rangle$$



$$k_1 = 0, \pi \equiv k_1^*$$

$$W^{\sigma\sigma'}_{2\pi\leftarrow 0}(k_1^*) = B^{-}_{(k_1^+,0)}(I)\, W^{+}_{\pi\leftarrow 0}(k_1^+)\, B^{--}_{(k_1^+,\pi)}(I)\, W_{\pi\leftarrow 0}(k_1^+)$$

$$= B^{-}_{(k_1^+,0)}(I)\, B^{--}_{(k_1^+,\pi)}(I)$$

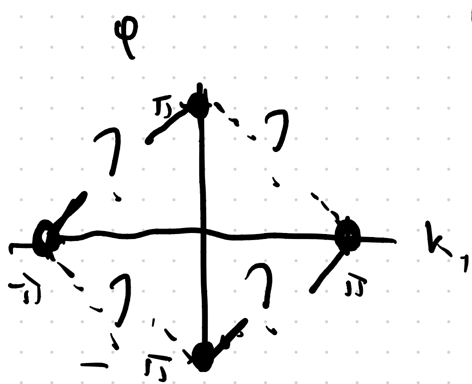
Since the sewing matrices are all $\propto$ identity

$k_1^* = 0$: $W_{2\pi \leftarrow 0}(k) = (-\sigma_0)(\sigma_0) = -\sigma_0$

↳ $(e^{i\pi}, e^{-i\pi})$ eigenvalues

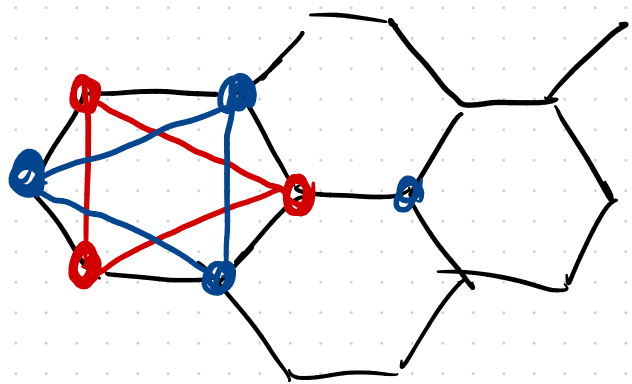
$k_1^* = \pi$ $W_{2\pi \leftarrow 0}(0) = (+\sigma_0)(+\sigma_0) = +\sigma_0$

↳ (e^{i0}, e^{i0}) eigenvalues



Kane-Mele: Spin-orbit coupling

$$\delta h(\vec{k}) = \lambda \left[\sin k_1 - \sin k_2 + \sin(k_2 - k_1)\right] \tau_z \sigma_z$$



Next-nearest neighbor spin-dependent hopping

Inversion symmetry

$$\tau_x\sigma_0\left(\delta h(\vec{k})\right)\tau_x\sigma_0 = -\lambda\left[\sin k_1 - \sin k_2 + \sin(k_2-k_1)\right]\tau_z\sigma_z$$
$$= \delta h(-\vec{k})$$

Time-reversal symmetry:

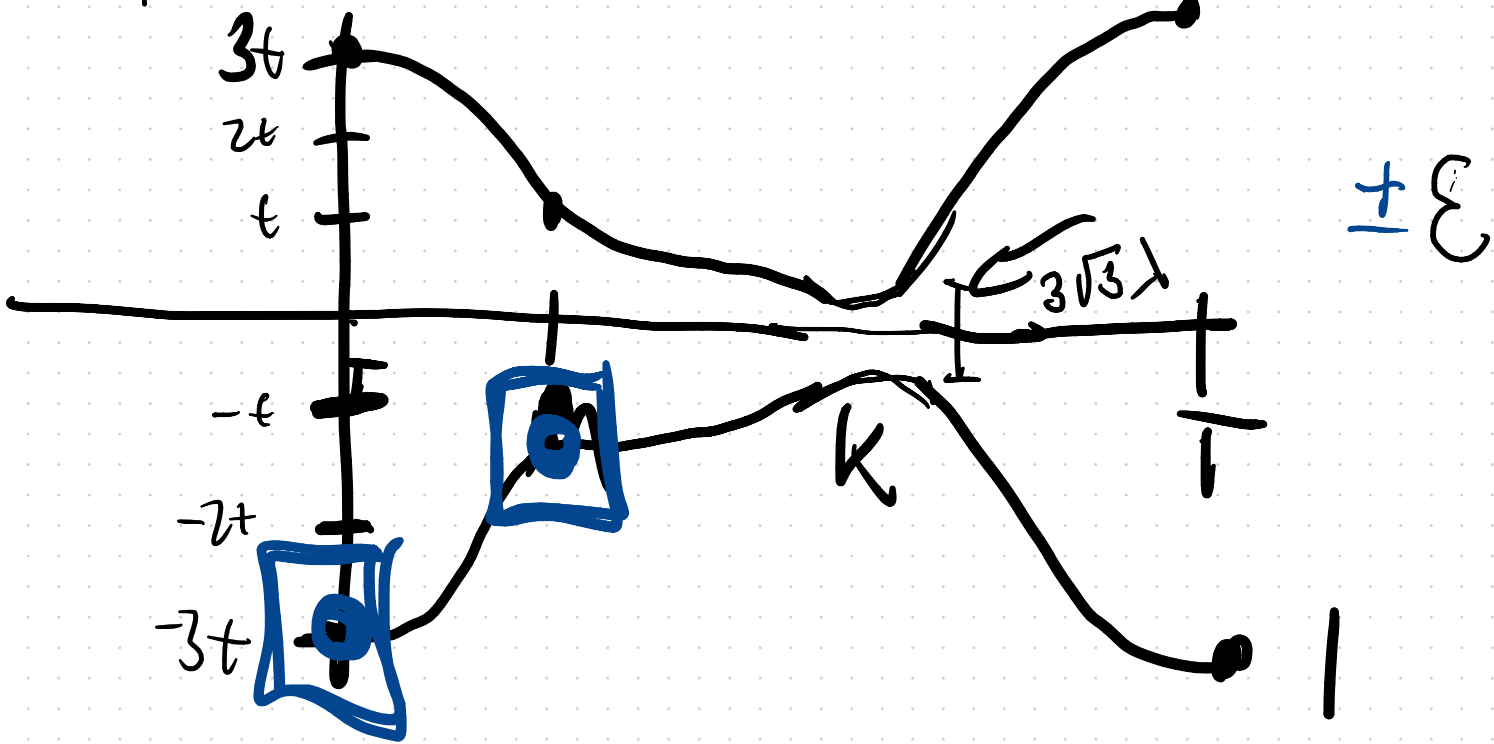
$$i\sigma_y\left[\delta h(k)\right]^*(-i\sigma_y) = -\delta h(k) = \delta h(-k)$$

Effect on the spectrum

$$\delta h(\Gamma) = 0$$
$$\delta h(M) = 0$$
$$\delta h(K) = \delta h\left(k_1 = \frac{4\pi}{3},\ k_2 = \frac{2\pi}{3}\right)$$

$$= \tau_z \sigma_z \lambda \left[\sin\frac{4\pi}{3} - \sin\frac{2\pi}{3} - \sin\frac{2\pi}{3} \right] = -\frac{3\sqrt{3}}{2}\lambda \tau_z \sigma_z$$

$h(k) + \delta h(k)$

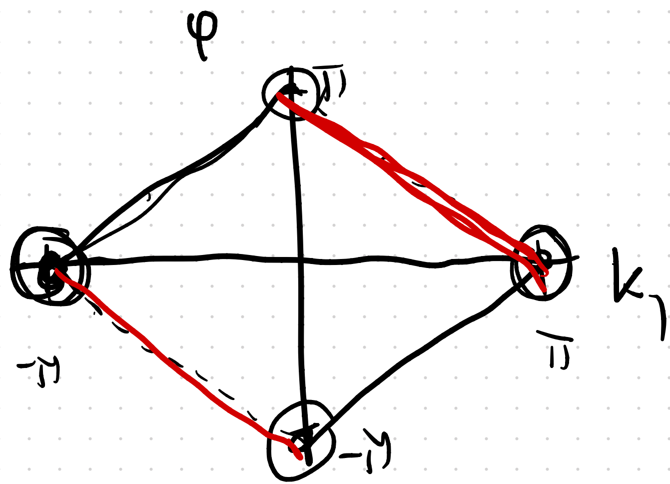


→ Wilson loop eigenvalues $W_{2\pi \leftarrow 0}(k_1)$ for

Kane-Mele hamiltonian

$C = 0$

no Chern number



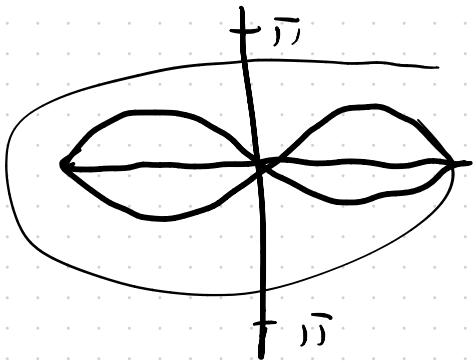
Degeneracy is enforced by time-reversal symmetry

"helical winding"

2D Time-reversal invariant topological insulator

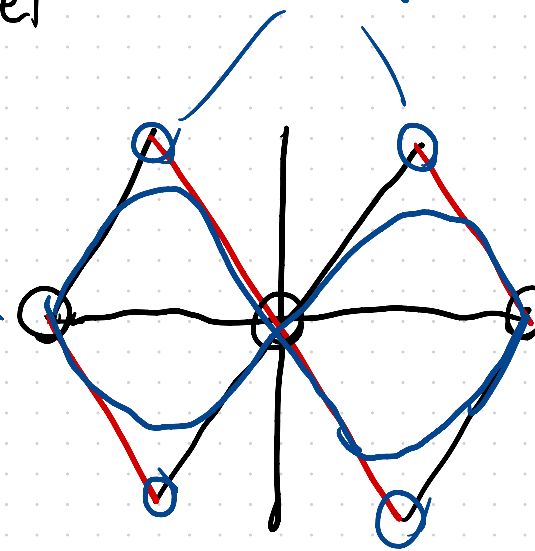
"Quantum Spin Hall Insulator"

ν - helical winding number

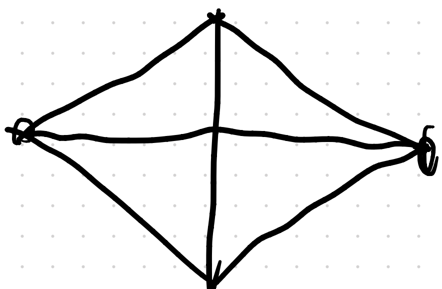


$\nu = 0$

not protected



$\nu = 2$



$\nu = 1$

ν mod 2 - the $\mathbb{Z}_2$ invariant

"Kane-Mele invariant"