Lecture 3

Announcements:

- Office Hrs start tomorrow 4pm - 5pm, Zoom link on contact info page of course website
- HW1 will be posted this evening, due 9/12

Last two lectures: Groups, subgroups, cosets, quotient groups, 1st Isomorphism theorem

One last point about quotient groups

$H \triangleleft G$, H a normal subgroup of G

$$G = H \cup Hg_1 \cup Hg_2 \cup \cdots \cup Hg_{n-1}$$

quotient group $G/H = \{H, Hg_1, Hg_2, \dots, Hg_{n-1}\}$

In some cases, there exists a homomorphism

$$i : G/H \to G$$

$$i(Hg_i) = g_i \in G$$

$$i(H) = E \in G$$

if i exists and is a group homomorphism then the set $K = \{E, g_1, g_2, \dots g_{n-1}\}$ forms a group, subgroup of G, isomorphic to G/H

then every $g \in G$ can be written as hk $h \in H$
$k \in K$

$$G = HK$$

if this is possible, we say $G = H \rtimes K$ G is the <u>semidirect product</u> of H with K

Example: The group of rigid transformations of 3D space

- rotations
- reflections
- translations

$\mathbb{E}(3)$ Euclidean group

$\mathbb{E}(3) \ni g = \{ R \mid \vec{v} \}$ Seitz symbol for g

$R \in O(3)$ rotation or reflection

$\vec{v} \in \mathbb{R}^3$ translation

Actson on points in space

$$g\vec{x} = R\vec{x} + \vec{v}$$

Multsplscatson:

$$g_1 = \{R_1 | \vec{v}_1\}$$

$$g_2 = \{R_2 | \vec{v}_2\}$$

$$(g_1 g_2)\vec{x} = g_1(g_2\vec{x}) = g_1(R_2\vec{x} + \vec{v}_2)$$

$$= R_1(R_2\vec{x} + \vec{v}_2) + \vec{v}_1$$

$$= R_1 R_2 \vec{x} + (\vec{v}_1 + R_1\vec{v}_2)$$

$$g_1 g_2 = \{R_1 R_2 | \vec{v}_1 + R_1\vec{v}_2\}$$

$g_2^{-1} = \{R_2^{-1} | -R_2^{-1} V_2\}$ $\quad g_2^{-1} g_2 = \{E | \vec{0}\}$

$\{E|\vec{0}\}$ is the identity.

- the group of translations $\left\{ \{E|\vec{v}\} \,|\, \vec{v} \in \mathbb{R}^3 \right\} = \mathbb{R}^3$

is a normal subgroup of $\mathbb{E}(3)$

check: $\{R|d\}^{-1}\{E|\vec{v}\}\{R|\vec{d}\}$

$= \{R^{-1}|-R^{-1}d\}\{R|\vec{v}+\vec{d}\}$

$= \{E|R^{-1}(v)\} \in \mathbb{R}^3$

$$\mathbb{R}^3 \triangleleft \mathbb{E}(3)$$

$$\{R|\vec{v}\} = \{E|\vec{v}\}\{R|\vec{0}\} \in [\mathbb{R}^3][O(3)]$$

so $$\mathbb{E}(3) = \mathbb{R}^3 \rtimes O(3)$$

If

$$\{R_1|V_1\}\cdot\{R_2|V_2\} = \{R_1R_2|V_1V_2\}$$

Direct product $(a, b)(c, d) = (ac, bd)$

Semidirect product $(ac, ba^{-1}da)$

How do we use groups in solid state physics.

Hamiltonian $H = \frac{p^2}{2m} + V(x)$

$$[x, p] = i\hbar$$

invariant under a group of transformations G

$$x \to x' = gx$$

$$p \to p' = g^{-1}p$$

$$\psi'(x) = \psi(g^{-1}x)$$

We can look for unitary operators U_g for each g that implement these transformations

$$|\psi'\rangle = U_g|\psi\rangle$$

$$U_g^\dagger x U_g = x'$$
$$U_g^\dagger \rho U_g = \rho'$$

we want $\varphi : g \to U_g$ to be a group homomorphism

we want $U_{g_1 g_2} = U_{g_1} U_{g_2}$

$$U_E = \mathbb{1}$$

$$U_{g^{-1}} = U_g^\dagger$$

We define: a (unitary) representation of a group G is a vector space V and a homomorphism

$$\rho : G \to U(V)$$

our group

unitary operators/matrices on V

$\rho(g)$ is the representative of g

Example: $SU(2)$ every element g of $SU(2)$ is specified by an axis $\hat{n}$ an an angle $\theta \in [0, 2\pi)$

$$(\hat{n}, \theta) \to \cos\frac{\theta}{2}\sigma_0 + i\sin\frac{\theta}{2}\, \hat{n}\cdot\vec{\sigma}$$

$\vec{\sigma} =$ 2×2 pauli matrices

σ_0 2×2 identity

<u>Defining representation</u> of $SU(2)$

But we also have the spin-I representation

$$(\hat{n}, \theta) \to e^{-i\theta\hat{n}\cdot\vec{I}}$$

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_y = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\frac{1}{\sqrt{2}}$$

$$L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Suppose we have a representation ρ of a group G on space V. We say that $W \subset V$ is an <u>invariant subspace</u> if $\rho(g)|w\rangle \in W$ for all $|w\rangle \in W$ and $g \in G$

Since $\rho(g)$ are unitary, if W is an invariant subspace, then so is $W^\perp = \{ |v\rangle \in V \mid \langle v|w\rangle = 0 \text{ for all } |w\rangle \in W\}$

Since $V = W \oplus W^{\perp}$ then if we write

$$\rho(g) = \begin{pmatrix} \rho_{11}(g) & \rho_{12}(g) & \cdots \\ \rho_{21}(g) & \ddots & \\ \vdots & & \ddots \end{pmatrix}$$

we can choose a basis for V such that

$$\rho(g) = \left(\begin{array}{c|c} \rho_W(g) & 0 \\ \hline 0 & \rho_{W^{\perp}}(g) \end{array}\right) \begin{array}{l} \} \text{ basis vectors for } W \\ \} \text{ basis vecters for } W^{\perp} \end{array}$$

ρ_W and $\rho_{W^{\perp}}$ are also representations of G

ρ_W is a represention of G on W

$$V = W \oplus W^{\perp}$$

$\rho_{W^\perp}$ is a representation of G on $W^\perp$

$$\rho(g) = \rho_W(g) \oplus \rho_{W^\perp}(g)$$

We say ρ is a <u>reducible</u> representation

A representation that is not reducible is called <u>irreducible</u>

(a representation is irreducible if the only invariant subspaces are $\{0\}$ and the entire space)

Note: Every group G has a special 1D irreducible representation

$$\rho(g) = \underline{1}$$

trivial representation

Example: Consider two spin-$\frac{1}{2}$ particles, with Hilbert space $V = \{ |\uparrow\uparrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle \}$

these transform in a representation

$$\rho((\hat{n},\theta)) = e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}_1} \otimes e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}_2}$$

the 1D subspace $W = \left\{ \frac{1}{\sqrt{2}} \left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \right\}$

is an invariant subspace - spin-0 singlet

$$W^{\perp} = \left\{ |\uparrow\uparrow\rangle, \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right), |\downarrow\downarrow\rangle \right\}$$

in the $W \oplus W^{\perp}$ basis

$$\rho([\hat{n},\theta]) \doteq \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & e^{-i\theta\hat{n}\cdot\vec{L}} \end{array}\right) \begin{array}{l} \}\, W \\ \}\, W^{\perp} \end{array}$$

Clebsch-Gordan coefficients - bases for invariant subspaces