Lecture 4

Reminders:
- HW 1 is now posted
- Office hours Wednesdays 4-5pm

Recap: representation ρ of a group G on a vector space V

$$\rho : G \to U(V)$$

homomorphism

A representation ρ is irreducible if the only invariant subspaces of V are $\{\vec{0}\}$ and V

If ρ is reducible, $\rho \cong \rho_1 \oplus \rho_2$

equivalent up to unitary transformations/changes of basis

Schur's Lemma

part I: We have a group G and two irreducible representations

$$\rho_1 : G \to U(V_1)$$
$$\rho_2 : G \to U(V_2)$$

If we have a linear map $A: V_1 \to V_2$

that satisfies $\boxed{A\rho_1(g) = \rho_2(g)A}$ for all $g \in G$

then either: $A = 0$ or A is invertible

Pf.: lets look at $\operatorname{Ker} A = \{v \in V_1 \mid Av = 0\}$

if $v \in \operatorname{Ker} A$ then $\rho_1(g)v \equiv w$

$$Aw = A\rho_1(g)v = \rho_2(g)Av = 0$$

if $v \in \operatorname{Ker} A$, so is $\rho_1(g)v \in \operatorname{Ker} A$

$\Rightarrow$ $\operatorname{Ker} A$ is an invariant subspace of ρ_1

but ρ_1 is irreducible $\Rightarrow$ $\operatorname{Ker} A = \begin{cases} \{\vec{0}\} \\ V_1 \to A \text{ is the zero matrix} \end{cases}$

if $\operatorname{Ker} A = \{\vec{0}\} \Rightarrow A$ is one-to-one

$$\left(\begin{array}{l} Av_1 = Av_2 \\ v_1 - v_2 \in \operatorname{Ker} A \end{array} \right)$$

Now let's consider $\operatorname{Im} A = \{ w \in V_2 \mid w = Av_1 \text{ for } v_1 \in V_1 \}$

is $\operatorname{Im} A \subset V_2$ an invariant subspace? Yes!

$w \in \operatorname{Im} A \quad w = Av_1$

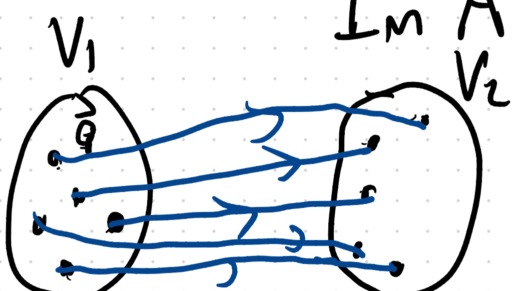
$$\rho_2(g)w = \rho_2(g)Av_1 = A[\underbrace{\rho_1(g)v_1}_{\text{some vector in } V_1}]$$

$$\rho_2(g)w \in \operatorname{Im} A$$

$$\operatorname{Im} A = \begin{cases} \boxed{\{\vec{0}\}} \to \operatorname{Ker} A = V_1 \Rightarrow A = 0 \\ V_2 \end{cases}$$

if $A \neq 0$ $\quad \operatorname{Ker} A = \{\vec{0}\}$ - one-to-one

$\operatorname{Im} A = V_2$ - surjective



$\to$ $\boxed{A \text{ is invertible}}$

$\boxed{\ker A = V_1 \Rightarrow A = 0}$

$$0 = Av = \begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \cdots & \vec{a}_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_n \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = \vec{a}_1 = 0$$

$$A \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{a}_2 = 0$$

Part 2.5) if $A\rho_1(g) = \rho_2(g)A$ for all $g \in G$

and A is invertible

then $\rho_2(g) = U\rho_1(g)U^\dagger$

so ρ_1 and ρ_2 are equivalent

$$A: V_1 \to V_2$$

$$A\rho_1(g) = \rho_2(g)A$$

taking Hermitian conjugates:

$$\rho_1^\dagger(g) A^\dagger = A^\dagger \rho_2^\dagger(g) \quad \text{for all } g \in G$$

$$\rho_1(g^{-1}) A^\dagger = A^\dagger \rho_2(g^{-1}) \quad \text{for all } g \in G$$

$\Rightarrow$ $A^\dagger : V_2 \to V_1$ satisfies our assumptions from Schur's lemma, $A^\dagger$ is invertible

$A^\dagger A : V_1 \to V_1$ and is invertible

$$A^\dagger A \rho_1(g) = A^\dagger \rho_2(g) A = \rho_1(g) A^\dagger A$$

$$[A^\dagger A, \rho_1(g)] = 0$$

Schur's Lemma part 2

$\rho_1 = \rho_2 = \rho$ irreducible

and $A: V_1 \to V_1$ $A\rho(g) = \rho(g)A$ for all $g \in G$

$$[A, \rho(g)] = 0$$

then if A is invertible, $A = \lambda \, Id$

identity

Pf: A is invertible (square) and so we know A has at least one eigenvector v and eigenvalue λ

$$(A - \lambda \, Id) = B$$

$[B, \rho(g)] = 0 \Rightarrow B$ is either invertible or 0

But B cannot be invertible, because

$$Bv = (A - \lambda \, Id) v = 0 \Rightarrow \ker B \ni v$$

$\Rightarrow B = 0$ by Schur's lemma

$$A = \lambda \, Id$$

This applies to QM when A is the Hamiltonian

$\{|\psi_i\rangle\}$ states transforming in some irreducible

irreducible representation of a symmetry group G

$$U_g \underline{|\psi_i\rangle} = \sum_j |\psi_j\rangle \rho_{ji}(g)$$

$$\underline{U_g^\dagger H U_g = H}$$

$$[H]_{ij} = \langle \psi_i | H | \psi_j \rangle$$

$$= \left[\langle \psi_i | U_g^\dagger \right] H \left[U_g | \psi_j \rangle \right]$$

$$= \sum_{kl} \rho_{ik}(g)^\dagger [H]_{kl}\, \rho_{lj}(g)$$

$\rho(g)[H] = [H]\rho(g)$ $\rightarrow$ Schur's lemma $H = E_n \delta_{ij}$

$\rightarrow$ States transforming in a irreducible representation of a symmetry group are degenerate

Ex: Hydrogen atom Hamiltonian (invariant under $SO(3)$)

$$\{ |n l m_z\rangle \mid m_z = -l, \dots, l \}$$

$$\langle n l m_z | H | n l m_z' \rangle = H_{m_z m_z'} \overset{\text{Schur's lemma}}{=} E_{nl}\, \delta_{m_z m_z'}$$

Spin - ℓ representation of SO(3)

$$\rho[(\hat{n},\theta)] \doteq e^{-i\theta\hat{n}\cdot\vec{J}_\ell}$$

$$[J_\ell^i, J_\ell^j] = i\,\varepsilon_{ijk} J_\ell^k$$

consider $d\theta$ infinitesimal

$$\rho[(\hat{n},d\theta)] \doteq 1 - i\,d\theta\,\boxed{\hat{n}}\,\underline{\vec{J}}$$

Letssay $\rho_1 = \eta_1 \oplus \eta_2 \oplus \cdots$

$\rho_2 = \sigma_1 \oplus \sigma_2 \oplus \cdots$

$$A : V_1 \to V_2$$

$$A : (W_1 \oplus W_2 \oplus \cdots) \to (T_1 \oplus T_2 \oplus \cdots)$$

$$A = \begin{array}{c|c} & \begin{array}{ccc} W_1 & W_2 & W_3 \cdots \end{array} \\ \begin{array}{c} T_1 \\ T_2 \\ T_3 \end{array} & \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \end{array}$$

$$\rho_1 = \eta_1 \oplus \eta_1 \oplus \eta_2 \oplus \cdots$$

$$\rho_2 = \eta_1 \oplus \eta_1 \oplus \cdots$$

Schur's lemma block by block

$$A\rho_1(g) = \rho_2(g) A$$

$$A \neq 0$$

$$[A^\dagger A, \rho_1(g)] = 0 \Rightarrow A^\dagger A = \lambda \, \mathrm{Id}$$

$$A^\dagger = \lambda A^{-1}$$

$$U = \frac{1}{\sqrt{\lambda}} A$$

$$U^\dagger = \frac{1}{\sqrt{\lambda}} A^\dagger = \sqrt{\lambda} A^{-1}$$

$$U^\dagger U = \mathbb{1}$$

$$\rho_1(g) = A^{-1} \rho_2(g) A$$

$$= \frac{1}{\sqrt{\lambda}} U^\dagger \rho_2(g) \sqrt{\lambda}\, U$$

$$= U^\dagger \rho_2(g) U$$

$$\rho_1 \simeq \rho_2$$

Character Theory: Lets us do the things:

- Lets us tell when two representations are the same
- Lets us tell when a representation is irreducible
- Lets us count irreducible representations..

The **character** χ_ρ of a representation ρ

$$\chi_\rho : G \to \mathbb{C}$$

$$\chi_\rho(g) = \mathrm{tr}[\rho(g)]$$

1. If the representations $\rho_1 \cong \rho_2$

$$U^\dagger \rho_1(g) U = \rho_2(g) \quad \text{for all } g \in G$$

$$\begin{aligned} \chi_{\rho_2}(g) &= \mathrm{tr}[\rho_2(g)] \\ &= \mathrm{tr}[U^\dagger \rho_1(g) U] \\ &= \mathrm{tr}[\rho_1(g)] = \chi_{\rho_1}(g) \end{aligned}$$

2. if $g_2 = g g_1 g^{-1}$ then $\chi_\rho(g_2) = \mathrm{tr}[\rho(g g_1 g^{-1})]$

$$= \mathrm{tr}[\rho(g)\rho(g_1)\rho^\dagger(g)]$$

$$= \chi_\rho(g_1)$$

$\Rightarrow$ Characters are constant on conjugacy classes

③ If $\rho_3 = \rho_1 \oplus \rho_2$

$$\rho_3(g) = \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right)$$

$$\chi_{\rho_3} = \chi_{\rho_1} + \chi_{\rho_2}$$