Lecture 5

Recap: Schur's Lemma if we have a group G, and two irreducible representations (irreps) $\rho_1 : G \to U(V_1)$
$\rho_2 : G \to U(V_2)$

then if we have a linear map $A : V_1 \to V_2$ such that $A\rho_1(g) = \rho_2(g)A$

then: ① $A = 0$
② $\rho_1 \cong \rho_2$, A is invertible, and in a basis where $\rho_1 = \rho_2$, $A = \lambda\, Id$
[$A = \lambda U$ U is a unitary change of basis]

Character of a representation $\chi_\rho : G \to \mathbb{C}$

$$\chi_\rho(g) = \mathrm{tr}(\rho(g))$$

① Characters are invariant under conjugation:

$$\chi_\rho(g^{-1} g_1 g) = \chi_\rho(g_1) \leftarrow \text{"Class functions"}$$

② $\rho_3 \cong \rho_1 \oplus \rho_2 \Rightarrow \chi_{\rho_3} = \chi_{\rho_1} + \chi_{\rho_2}$

③ If $\rho_1 \cong \rho_2$, then $\chi_{\rho_1} = \chi_{\rho_2}$

$$U^\dagger \rho_1(g) U = \rho_2(g)$$

$$\left(④ \quad \rho_3 = \rho_1 \otimes \rho_2 \qquad \chi_{\rho_3}(g) = \chi_{\rho_1}(g)\chi_{\rho_2}(g) \right)$$

Suppose $\rho_1 : G \to U(V_1)$

$\rho_2 : G \to U(V_2)$ are irreps

and A be any map $A : V_1 \to V_2$

then $\boxed{A_G = \sum_{g \in G} \rho_2(g^{-1}) A \rho_1(g)}$ satisfies

$$\rho_2(g) A_G = A_G \rho_1(g)$$

pf: $\rho_2(g) A_G = \rho_2(g) \sum_{g' \in G} \rho_2(g'^{-1}) A \rho_1(g')$

$$= \sum_{g' \in G} \rho_2(g g'^{-1}) A \rho_1(g')$$

let $\quad g'' = g' g^{-1} \qquad\qquad g g'^{-1} = (g' g^{-1})^{-1} = g''^{-1}$

$$= \sum_{g'' \in G} \rho_2(g''^{-1}) A \rho_1(g'' g)$$

$$= \left[\sum_{g'' \in G} \rho_2(g''^{-1}) A \rho_1(g'') \right] \rho_1(g)$$

$$= A_G \rho_1(g)$$

so by Schur's lemma, either $\rho_1 \not\cong \rho_2 \Rightarrow A_G = 0$

or if $\rho_1 = \rho_2$, $A_G = \lambda \, \mathrm{Id}$

let's apply this when $A = E_{ij}$

$$[E_{ij}]^{\mu\nu} = \delta_{i\mu}\delta_{j\nu}$$

$$\begin{array}{c} ij\text{-th} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 \cdots & 1 & 0 & 0 \; 0 \\ \vdots & 0 & 0 & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix} \end{array}$$

$$[E_{ij}]_G^{\alpha\beta} = \sum_{g \in G} \left[\rho_2(g^{-1}) E_{ij} \, \rho_1(g)\right]^{\alpha\beta}$$

$$= \sum_{\mu,\nu} \sum_{g\in G} [\rho_2(g^{-1})]^{\alpha\mu} [E_{ij}]^{\mu\nu} [\rho_1(g)]^{\nu\beta}$$

$$= \sum_{g\in G} [\rho_2(g^{-1})]^{\alpha i} [\rho_1(g)]^{j\beta} = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ \lambda \delta_{\alpha\beta} & \text{if } \rho_1 = \rho_2 \end{cases}$$

So lets find λ when $\rho_1 = \rho_2$

$$tr([E_{ij}]_G) = \sum_{\alpha\beta} [E_{ij}]_G^{\alpha\alpha}$$

$$= \sum_{g\in G} \sum_{\alpha} [\rho_1(g^{-1})]^{\alpha i} [\rho_1(g)]^{j\alpha}$$

$$= \sum_{g \in G} \left[\rho_1(g) \rho_1(g^{-1}) \right]^{ji}$$

$|G|$ = # of elements in G

"Magic formula" "Wonderful orthogonality theorem" "Schur Orthogonality Relations"

$$= \sum_{g \in G} \left[\rho_1(E) \right]^{ji}$$

$$= \sum_{g \in G} \delta_{ij} = |G| \delta_{ij} = \lambda \, tr(\delta_{\alpha\beta})$$

$$= \lambda \dim \rho_1$$

$$\rightarrow \boxed{\sum_{g \in G} \left[\rho_2(g^{-1}) \right]^{\alpha i} \left[\rho_1(g) \right]^{j\beta} = \begin{cases} 0 & \text{if } \rho_1 \neq \rho_2 \\ \frac{|G|}{\dim \rho_1} \delta_{\alpha\beta} \delta_{ij} & \text{if } \rho_1 = \rho_2 \end{cases}}$$

If we set $\alpha = i$ $\beta = j$ and sum over α, β

$$\sum_{g \in G} \chi_{\rho_2}(g^{-1}) \chi_{\rho_1}(g) = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ \frac{|G|}{\cancel{\dim \rho_1}} \cancel{\dim \rho_1} & \text{if } \rho_1 \cong \rho_2 \end{cases}$$

$$\frac{1}{|G|} \sum_{g \in G} \chi^*_{\rho_2}(g) \chi_{\rho_1}(g) = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ 1 & \text{if } \rho_1 \cong \rho_2 \end{cases}$$

$$\overset{\shortparallel}{\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle}$$

– Characters for irreps are orthonormal under this inner product lets say $\rho = \rho_1 \oplus \rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \ldots$

$\rho_1, \rho_2, \rho_3 \ldots$ are irreps

$$\chi_\rho = \underbrace{\chi_1}_{} + \chi_1 + \chi_2 + \chi_3 + \ldots$$

$$\langle \chi_{\rho_1}, \chi_\rho \rangle = \langle \chi_{\rho_1}, \chi_{\rho_1} \rangle + \langle \chi_{\rho_1}, \chi_{\rho_1} \rangle + \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle + \langle \chi_{\rho_1}, \chi_{\rho_3} \rangle + \ldots$$

$$= 2$$

$$\langle \chi_{\rho_2}, \chi_\rho \rangle = 1$$

$$\langle \chi_{\rho_3}, \chi_\rho \rangle = 1 \ldots$$

$$\chi_\rho = \sum_{\substack{\text{irreducible} \\ \text{reps } \rho_i}} n_i \chi_{\rho_i}$$

$$n_i = \langle \chi_{\rho_i}, \chi_\rho \rangle$$

multiplicity of ρ_i in the decomposition of ρ

Example: The group D_2 from HW1

$$D_2 = \{E, C_{2x}, C_{2y}, C_{2z}\}$$

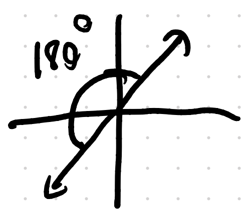
point group 222

Defining matrix representation

$$\rho_V(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \rho_V(C_{2x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\rho_V(C_{2y}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \rho_V(C_{2z}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a \\ \boxed{\begin{matrix} 0 \\ 0 \end{matrix}} \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$$



$$\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$$

all matrices diagonal $\Rightarrow$ multiplication is commutative

abelian group

abelian group $\quad g g_1 g^{-1} = g g^{-1} g_1 = g_1$

$\downarrow$

every element is its own conjugacy class

Constructing the character table

	E	C_{2x}	C_{2y}	C_{2z}
A	1	1	1	1
B_1	1	1	-1	-1
B_2	1	-1	1	-1
B_3	1	-1	-1	1

$\rho_V = B_1 \oplus B_2 \oplus B_3$

For 1D representations the characters are
the representation matrices