Brief overview of electroweak theory and the standard Model

In order to understand the peculiar form for the neutral current, and many other questions, some of which are listed below, we will briefly go over the electroweak theory in the standard model.

- Why is electron massive, but neutrino massless in the SM ?
- Since we now know that neutrino is massive, how can SM be extended to explain this ?
- Is ν it's own antiparticle ? How would a Dirac ν gain its mass ? How would a Majorana ν gain its mass ?
- How is the Higgs particle responsible for particle's mass ?

We need to discuss the topics of gauge invarian spontaneous symmetry breaking, Goldstone bosons, Higgs mechanism , etc .

Equations of motion and Euler-Lagrange Eq.

For a given Lagrangian density $\mathcal{L}$, the equation of motion is obtained from the Euler-Lagrange Eq., namely

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial\psi/\partial x_\mu)}\right)-\frac{\partial \mathcal{L}}{\partial \psi}=0$$

1) For spin-0 free particle

$$\mathcal{L}_0^{K-G}=\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{1}{2}m^2\phi^2$$

setting $\psi=\phi$, the Euler-Lagrange Eq. gives

$$(\partial_\mu\partial^\mu+m^2)\phi=0$$

which is the Klein-Gordon Eq.

2) For spin-½ free particle

$$\mathcal{L}_0^{Dirac}=\psi(i\gamma^\mu\partial_\mu-m)\psi$$

setting $\psi=\bar{\psi}$, the Euler-Lagrange Eq. gives
$(i\gamma^\mu\partial_\mu-m)\psi=0$, which is the Dirac Eq.

3) For spin-1 photon

$$\mathcal{L}_0^{photon}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-j^\mu A_\mu\ , \text{ where } F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu$$

setting $\psi=A_\mu$, the Euler-Lagrange Eq. gives
$\partial_\mu F^{\mu\nu}=j^\nu$, which is the Maxwell Eq.

Gauge invariance

We start by considering a free spin-½ particle, which is described by the Lagrangian density

$$\mathcal{L}_0^{Dirac} = \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi$$

We now consider the consequences of invariance of $\mathcal{L}$ (and equation of motion) under unitary transformation, $SU(n)$.

1) Global $U(1)$ transformation:

$$\psi'(x) = e^{-iqw}\,\psi(x)$$

q is a fixed constant specifying the $U(1)$ coupling strength. w is a parameter specifying the amount of 'rotation angle' in the $U(1)$ transformation.

When w is independent of $x = (t, \vec{x})$, it is called a 'global' $U(1)$ transformation. Otherwise, it is a 'local' $U(1)$ transformation.

For a global $U(1)$ transformation, we have

$$\psi'(x) = e^{-iqw}\,\psi(x)$$
$$\bar{\psi}'(x) = e^{iqw}\,\bar{\psi}(x)$$
$$\partial_\mu \psi'(x) = e^{-iqw}\,\partial_\mu \psi(x)$$

Hence, the field gradient, $\partial_\mu\psi(x)$, transform the same as the field ψ.
We have $\mathcal{L}_0'^{Dirac} = \mathcal{L}_0^{Dirac}$,
and $\mathcal{L}_0^{Dirac}$ is invariant under the global $U(1)$ transformation

2) Local $U(1)$ transformation

For 'local' transformation, the "rotation angle", ω now depends on space-time:

$$\psi'(x) = e^{-iq\omega(x)}\psi(x)$$

We then have

$$\partial_\mu \psi'(x) = e^{-iq\omega(x)}(\partial_\mu \psi(x) - iq(\partial_\mu \omega(x))\psi(x))$$

Hence, the field gradient transforms differently from the field under local $U(1)$ transformation.

$$\mathcal{L}_0' = \mathcal{L}_0 + q\bar{\psi}\gamma^\mu\psi\partial_\mu\omega = \mathcal{L}_0 + j^\mu\partial_\mu\omega$$

where $j^\mu = q\bar{\psi}\gamma^\mu\psi$ is the vector current, which is invariant under local $U(1)$ transformation since it does not contain field gradient:

$$j^{\mu\prime} = q\bar{\psi}'\gamma^\mu\psi' = j^\mu$$

To restore local $U(1)$ invariance, one adds a new term (interaction term) to the $\mathcal{L}_0$ to obtain $\mathcal{L}_1$

$$\mathcal{L}_1 = \mathcal{L}_0 - j^\mu A_\mu$$

A_μ is a vector field to contract with the vector current j. It is also called a 'gauge field'. Under local $U(1)$ transformation, A_μ transforms in such a way that $\mathcal{L}_1$ becomes invariant under local $U(1)$ transf.:

$$A_\mu' = A_\mu + \partial_\mu\omega$$

We now have

$$\mathcal{L}_1' = \mathcal{L}_0' - j'^{\mu} A_\mu' = (\mathcal{L}_0 + j^\mu \partial_\mu w) - j^\mu (A_\mu + \partial_\mu w)$$
$$= \mathcal{L}_0 - j^\mu A_\mu = \mathcal{L}_1$$

Hence, $\mathcal{L}_1$ is now invariant under local U(1) transf.

$$\mathcal{L}_1 = \bar\psi (i\gamma^\mu \partial_\mu - m)\psi - q\bar\psi\gamma^\mu\psi A_\mu$$
$$= \bar\psi (i\gamma^\mu D_\mu - m)\psi$$

where $D_\mu = \partial_\mu + iqA_\mu$ is called the 'covariant' derivat

Since $D_\mu' \psi'(x) = e^{iqw(x)} D_\mu \psi(x)$,
we see that the covariant derivative of the field

$D_\mu \psi(x)$ transforms the same way as $\psi(x)$

and $\mathcal{L}_1 = \bar\psi (i\gamma^\mu D_\mu - m)\psi$ is invariant under local U(

One can add other terms to the $\mathcal{L}$ built up from the gauge field A_μ and its derivative, provided the U(1) local gauge invariance is maintaine

Since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant under the transformation $A_\mu \longrightarrow A_\mu + \partial_\mu w$

$$F_{\mu\nu}' = \partial_\mu A_\nu' - \partial_\nu A_\mu' = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$$

The Lorentz scalar $F_{\mu\nu}F^{\mu\nu}$ is also U(1) – invariant.

Hence, $\mathcal{L} = \bar\psi (i\gamma^\mu D_\mu - m)\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$ is obtaine

It is important to note that a mass term $\frac{1}{2} m^2 A_\mu A^\mu$ is not allowed, since it is not invariant under $A_\mu \to A_\mu + \partial_\mu w$

Therefore, GAUGE FIELD IS MASSLESS !

3) Global SU(n) transformation, $n > 1$

Consider the case when there are n different Dirac fields, ψ^a, $a = 1, \cdots n$.
where 'a' denotes internal degree of freedom, such as 'isospin' or 'color'.

Under SU(n) transformation,

$$\psi^a \longrightarrow \psi'^a = U^a{}_b \psi^b \quad \text{(summed over b)}$$

$U^a{}_b$ is the element of an $n \times n$ unitary matrix U.

$$U = \exp[-i g w_i T_i] \quad \text{(summed over } i\text{)}$$

T_i is $n \times n$ matrix called 'generator' of SU(n) tra
$i = 1, \cdots N$, where $N = n^2 - 1$ is the number of generator

$[T_i, T_j] = i f_{ijk} T_k$ are the commutation relations for the generators

Examples: SU(2), $\psi : \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$ are the two weakisosp states

$n = 2. \Rightarrow N = 3$, $T_i = \tau_i/2$ (Pauli matric

SU(3), $\psi : \begin{pmatrix} \psi_R \\ \psi_B \\ \psi_G \end{pmatrix}$ are the three color states

$n = 3 \Rightarrow N = 8$, $T_i = \lambda_i/2$ (Gell-Mann matrices)

The free-field Lagrangian density is

$$\mathcal{L}_0 = \sum_{a=1}^{n} \bar{\psi}_a (i\gamma^\mu \partial_\mu - m)\psi^a$$

4) local SU(n) transformation, $n > 1$

In this case, $U = \exp[-ig\, w_i(x)\, T_i]$, and U is a function of space-time, x.

Then, $\mathcal{L}_0' = \mathcal{L}_0 + \bar{\psi}\, i\gamma^\mu (U^+ \partial_\mu U)\psi$

$$\psi' = U\psi, \quad \partial_\mu \psi' = U(\partial_\mu \psi) + (\partial_\mu U)\psi$$

Again, the field ψ and the field gradient $\partial_\mu \psi$ transform differently.

Introduce A_μ ($n \times n$ matrix) whose elements are gauge fields $A_{j\mu}$ ($j = 1 \to (n^2-1)$):

$$A_\mu = A_{j\mu} T_j \quad \text{(summed over } j\text{)}$$

Let the gauge fields transform under local SU(n) as $A_\mu' = \frac{i}{g}(\partial_\mu U)U^+ + U A_\mu U^+$

Let $\mathcal{L}_1 = \mathcal{L}_0 - g\bar{\psi}\gamma^\mu A_\mu \psi$

then, $\mathcal{L}_1 = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$, where $D_\mu = \partial_\mu + igA_\mu$

and $\mathcal{L}_1$ is invariant under local U(n) transformation

Considering infinitesimal transformation U, the expressi

$$A_\mu' = \frac{i}{g}(\partial_\mu U)U^+ + U A_\mu U^+$$

becomes

$$A_{i\mu}' = A_{i\mu} + \partial_\mu w_i - g f_{ijk} A_{j\mu} w_k$$

where

$$[T_i, T_j] = i f_{ijk} T_k$$

The gauge field also gives the following contribution to the Lagrangian:

$$\mathcal{L}_G = -\frac{1}{4} F_{\mu\nu}^i F_i^{\mu\nu}$$

where

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i - g f_{ijk} A_\mu^j A_\nu^k$$

The 'extra' term $g f_{ijk} A_\mu^j A_\nu^k$, which reflects the non-abelian nature of $SU(n)$, is required to ensure $\mathcal{L}_G$ be invariant under local $SU(n)$ transformatio

(since $A_\mu^{i\prime} = A_\mu^i + \partial_\mu w^i - g f_{ijk} A_{j\mu} w_k$)

The total Lagrangian is

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_G$$

where

$$\mathcal{L}_1 = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$
$$= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - g A_{i\mu}\bar{\psi}\gamma^\mu T_i \psi$$

$$\mathcal{L}_G = -\frac{1}{4} B_{\mu\nu}^i B_i^{\mu\nu} + \frac{g}{2} f_{ijk} B_{\mu\nu}^i A_j^\mu A_k^\nu - \frac{g^2}{4} f_{ijk} f_{ilm} A_{j\mu} A_k A_l^\mu A_m$$

$$(B_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i)$$

Quantum Chromodynamics (QCD), which is the theory for strong interaction, is obtained by requiring local SU(3) symmetry. The internal symmetry refers to the three 'colors' degrees of freedom for quarks. The QCD Lagrangian is written as

$$\mathcal{L}_{QCD} = \sum_{A=1}^{N_f} \bar{\psi}^A (i\gamma^\mu D_\mu - m_A)\psi^A - \frac{1}{4} F^i_{\mu\nu} F_i^{\mu\nu}$$

$N_f = 6$ for the 6 quark flavor, $i = 1 \to 8$ for the 8 (n^2-1, where $n=3$) gluon fields.

$D_\mu = \partial_\mu + i g_s A_\mu$ ($A_\mu = A_{j\mu} T_j$) are the 3×3 covariant derivatives.

$$\psi_A = \begin{pmatrix} \psi_R^A \\ \psi_B^A \\ \psi_G^A \end{pmatrix}$$ contains three color components for each quark flavor A

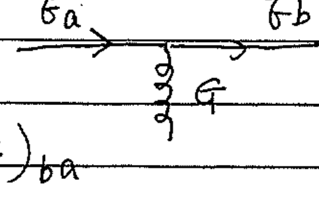
Note that $\mathcal{L}_{QCD}$ is flavor independent, except that m_A can be flavor dependent.

The gluon field tensor $F^i_{\mu\nu}$ is given as

$$F^i_{\mu\nu} = \partial_\mu G^i_\nu - \partial_\nu G^i_\mu - g_s f_{ijk} G^j_\mu G^k_\nu$$

From the $\bar{\psi} i\gamma^\mu D_\mu \psi$ term in $\mathcal{L}_{QCD}$, we obtain the quark-gluon coupling term as

$$-g_s \bar{\psi}(x) \gamma^\mu \frac{\lambda_i}{2} \psi(x) G_{i\mu}(x)$$



or the coupling vertex: $-i g_s \gamma_\mu \left(\frac{\lambda_i}{2}\right)_{ba}$

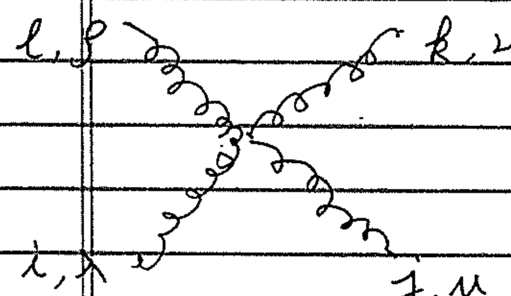
The gluon kinetic term

$$\mathcal{L}_G = -\frac{1}{4} F^i_{\mu\nu} F_i^{\mu\nu} = -\frac{1}{4} G^i_{\mu\nu} G_i^{\mu\nu} + \frac{g_s}{2} f_{ijk} G^i_{\mu\nu} G^\mu_j G^\nu_k - \frac{g_s^2}{4} f_{ijk} f_{ilm} G_{j\mu} G_{k\nu} G^\mu_l G^\nu_m$$

where $G^i_{\mu\nu} = \partial_\mu G^i_\nu - \partial_\nu G^i_\mu$

From $\mathcal{L}_G$, we find the three-gluon coupling term, as well as the four-gluon coupling term. These self-coupling terms reflect the non-abelian nature of SU(3) gauge symmetry.

For the four-gluon coupling, we have



$$-i g_s^2 [f_{ijm} f_{klm} (g_{\lambda\nu} g_{\mu\rho} - g_{\mu\nu} g_{\lambda\rho}) + f_{ikm} f_{jlm} (g_{\lambda\mu} g_{\nu\rho} - g_{\mu\nu} g_{\lambda\rho}) + f_{kjm} f_{ilm} (g_{\lambda\nu} g_{\mu\rho} - g_{\lambda\mu} g_{\nu\rho}$$

where i, j, k, l are color indices, and λ, μ, ν, ρ are the Lorentz indices for the metric tensor g

It seems natural to use SU(2) gauge symmetry to describe weak interaction. For example, $\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$ form SU(2) weak-isospin SU(2) doublet. However, the gauge bosons $W^\pm$ are massive, contrary to the requirement of massless gauge boson. How to solve this prob