

The quantum Hall effect: general considerations¹

We have so far dealt separately with 1D and 2D systems. But in a magnetic field, a system of charged particles (e.g. electrons) essentially has its dimensionality reduced by one, since the paths are bent into circles. At the classical level we see the effects of this even in 3D systems, e.g. in cyclotron resonance in metals. At the quantum level, the effects are more spectacular because the closed orbits now become quantized. However, in 3D metals the effects of this are somewhat blurred by the third dimension, and the only reason we can see anything interesting is that (e.g.) “extremal” regions of the Fermi surface tend to contribute an anomalously large amount; this is what leads to the well-known dHvA, Shubnikov-de Haas and other effects in bulk 3D metals. The effect of quantization of closed orbits comes out much more strikingly for systems which start off genuinely “2D”; however, even given this condition the requirements to see something interesting are quite stringent. We need

- (a) a scattering time which is much larger than the inverse of the cyclotron resonance frequency $\omega_c \equiv eB/m$
- (b) a mean free path much larger than the “magnetic length” (see below) defined by $l_M \sim (\hbar/eB)^{1/2}$
- (c) temperatures low enough that $k_B T \lesssim \hbar\omega_c$
- (d) a coverage (number of electrons per unit area) n such that $n \sim l_M^{-2} \sim eB/\hbar$

Since (at least until recently) the maximum attained value of B was around 10 T, the *maximum* value of n which can be tolerated is of order 10^{12} cm^{-2} .

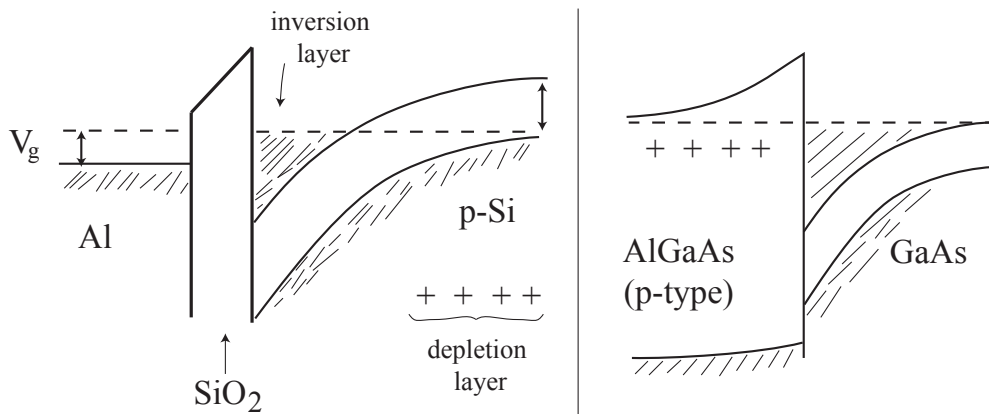
As already briefly discussed in lecture 3, the main types of system used for QHE experiments are

- (a) Si MOSFET’s, and
- (b) GaAs-AlGaAs heterostructures. (refs.: Prange et al. section I.8, II, VI, Yoshioka section I.1)

Si MOSFET’s:

the width of the *inversion* layer is typically $\sim 25\text{-}50 \text{ \AA}$ (width of the *depletion* layer is much greater). The effective mass of an electron at the bottom of the conduction band of Si is $0.2m_e$ (with six different “valleys” in bulk, often reduced to 2 near a surface) (note lattice has cubic symmetry) so the energy of the first “transverse” excited state is $\sim 100 \text{ K}$, and at the temperatures of interest the system may be regarded as 2D. Typical areal densities (controlled by the gate voltage) are in the range $10^{11}\text{-}10^{12} \text{ cm}^{-2}$, so that (taking the spin and “valley” degeneracy into account) the Fermi energy is in the range

¹R.E. Prange, S.M. Girvin, *The Quantum Hall Effect* (2nd edition), Springer 1990; D. Yoshioka, *The Quantum Hall Effect*, Springer 2002; J. Jain, *Composite Fermions*, Cambridge University Press, 2007.



4-40 K. This is very small compared to the width of the conduction band, so that the effective-mass approximation for the 2D motion should be very good. The mobility of electrons in the inversion layer seems to be limited mostly by scattering by the ionized acceptors in the depletion layer, and is usually not more than $\sim 40000 \text{ cm}^2/\text{V sec}$. The main advantage of the Si MOSFET system is that the areal density may be easily controlled by changing the gate voltage ($\epsilon \approx 12$ for Si, $m^*/m \sim 0.2$).

GaAs-GaAlAs heterostructures:

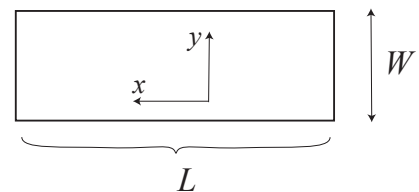
width of inversion layer $\sim 100 \text{ \AA}$ (again, width of “depletion” layer is much greater). Effective mass of electron in GaAs conduction band $\sim 0.07m_e$ (with only one “valley”). Generally, parameters similar in order of magnitude to Si MOSFET case. But mobility can be considerably greater (since the donors are far separated from the inversion region), up to $\sim 3 \times 10^7 \text{ cm}^2/\text{V sec}$. ($\epsilon \approx 11$).

[Also: electrons on surface of liquid ^4He (but $n \lesssim 10^8 \text{ cm}^{-2} \Rightarrow T_F \lesssim 2 \text{ mK}$)]

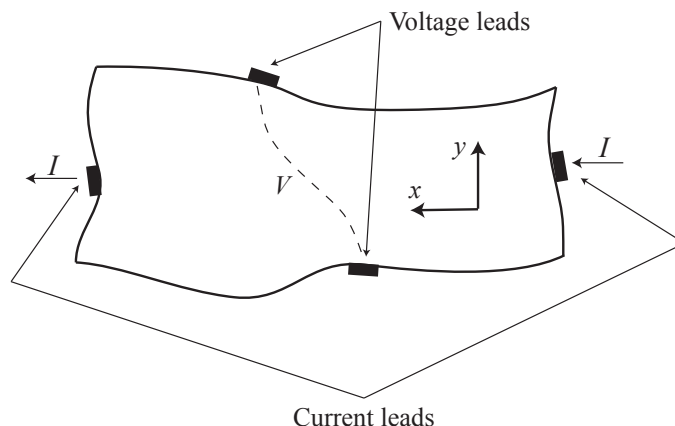
Conductances etc. in 2D:

Fundamental definitions: Consider a rectangular 2D block of material, length L , width W , then can define *resistance* R and *conductance* Σ tensors by

$$V_i \equiv \sum_j R_{ij} I_j, \quad I_i \equiv \sum_j \Sigma_{ij} V_j \quad (1)$$



Hence Σ is matrix inverse of R . Note carefully that (e.g.) R_{11} means the ratio of V_1 to I_1 with $I_2 = 0$, while Σ_{11} means the ratio of I_1 to V_1 with $V_2 = 0$; these two quantities are not necessary inverses! In particular, note that since e.g. $R_{xx} = \Sigma_{yy}/\det \Sigma$, it is perfectly possible to have R_{xx} and Σ_{xx} simultaneously zero (provided that Σ_{xy} and R_{xy} are nonzero).



Resistivity \equiv resistance (tensor) \times cross-sectional “area” (length in 2D) \perp to current flow / length over which voltage drops. In 3D $\dim \rho \sim \dim (RL)$, but in 2D resistance and resistivity have *same* dimension, and in fact $R_{\square} = \rho$ (as a tensor relation). For the “Hall resistance” things are even simpler: if magnetic field is out of page, R_H is defined to be \sim voltage drop across y -dimension (W) / current flowing in x -direction (with y -direction open-circuited so $I_y = 0$). Evidently in this case $R_H \equiv \rho_H$, with no dependence on L or W . (By contrast, $R_{xx} = (L/W)\rho_{xx}$). Generally, in the QHE it is much more convenient to discuss “ances” rather than “ivities”.

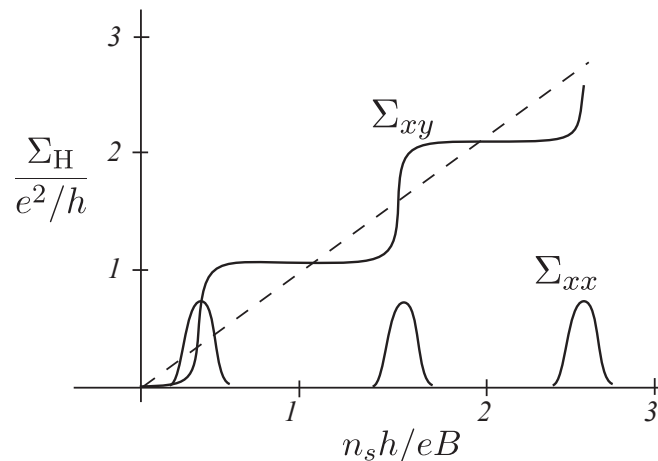
We will see, below, that the quantum Hall states (integer or fractional) are characterized by the property $\sigma_{xx} = 0$, $\sigma_{xy} = \text{const}$. States of this kind have a remarkable property: under very wide conditions the Hall *conductance* $\Sigma_H \equiv I/V$ measured in any four-terminal setup with the topology shown on the figure is independent of the geometry and equal to σ_{xy} ! This follows simply from $\text{div } \mathbf{j} = 0$ and $V = -\int \mathbf{E} \cdot d\mathbf{s}$, since $I = \int j_{\perp} ds = \sigma_{xy} \int \mathbf{E} \cdot d\mathbf{s} = \sigma_{xy} V \Rightarrow \Sigma_{\perp} \equiv I/V = \sigma_{xy} = R_H^{-1}$. So the “Hall” resistance is actually a special case of a more general four-terminal resistance.

The quantum Hall effect: phenomenology

The initial observations (von Klitzing et al., 1980) were that when the areal density of carriers n_s is held fixed and the magnetic field B varied (or vice versa) there are a number of *plateaux* in the Hall resistance, corresponding to value h/ne^2 where n is an integer. Over the length of each plateau the *longitudinal* resistance is zero within experimental error. Thus $\Sigma_H \equiv \Sigma_{xy} \equiv 1/R_H$ and $\Sigma_{xx} = 0$, and we get the graph plotted in terms of conductances, as shown.² The plateaux are centered on values of $n_s h/eB$ which correspond to integers. This is the *integral quantum Hall effect* (IQHE).

Subsequently (Tsui et al., 1982) it was discovered that there exist also some rational

²For historical reasons experimentalists conventionally plot R_H versus B , so that the graphs, while qualitatively similar to the one shown, do not possess the periodicity shown along either axis.



fractional values of $n_s h / eB$ ($\equiv \nu$), around which Hall plateaux can be centered; the corresponding value of the Hall conductance is $\nu(e^2/h)$. Almost without exception, the value of ν and which this “fractional quantum Hall effect” (FQHE) occurs are *fractions with odd denominators* (originally $1/3$, than $1/5$, $2/5$, $4/7$...); the one definitively known counterexample³ (as of October 2009) is $\nu = 5/2$, with some evidence also for $\nu = 7/2$ and possibly $\nu = 19/8$. The general behavior of the FQHE is similar to that of the IQHE, but it seems to be less robust against the effects of temperature and impurity scattering.

As far as is currently known, the integral (or rational-fractional) value of Σ_H in units of e^2/h on the plateaux is *exact*, and in fact it now forms a basis for metrology (see Prange et al., ch. II). This is at first sight very surprising, since not only variations in material properties such as the effective mass but even deviations from perfection in the geometry ought to be very large compared to 1 part in 10^8 ; nevertheless, the measured values of R_H seem to be independent of all such variations to about this accuracy!

As we shall see in a moment, there is absolutely no mystery about the fact that at exactly integral values of the “filling factor” $\nu \equiv n_s h / eB$ the Hall resistance in units of h/e^2 is exactly $1/\nu$; the mystery is why this result is maintained over a finite range of ν . Also, it appears that no “naive” theory can explain the appearances of plateaux at non-integral values of ν (the FQHE).

Classical considerations⁴

Let’s start by considering the simple problem of the motion of a free electron in an external electric field \mathbf{E} and magnetic field \mathbf{B} , in general not mutually parallel. Newton’s

³The phenomena which occur at $\nu = 1/2$, though very interesting, are not examples of the FQHE.

⁴Yoshioka, section 2.1.2.

equation is

$$m \frac{d^2 \mathbf{r}}{dt^2} = e\mathbf{E} + e\mathbf{v} \times \mathbf{B} \quad (2)$$

with the general solution

$$\mathbf{r}(t) = \mathbf{v}_0 t + \mathbf{r}_0(t) \quad (3)$$

where

$$\mathbf{v}_0 \equiv \mathbf{E} \times \mathbf{B}/B^2, \quad \mathbf{r}_0(t) = r_0(\cos \omega_c t, \sin \omega_c t, 0), \quad \omega_c \equiv eB/m \quad (4)$$

In words, the motion of the electron is a superposition of a steady drift in the direction perpendicular to both \mathbf{E} and \mathbf{B} , with velocity E/B , and a “cyclotron” (circular) motion of arbitrary amplitude around the magnetic field.

It is clear that this solution can be generalized to a situation when the “field” $\mathbf{E} \equiv -\nabla V$ acting on the electron is varying in space (but the magnetic field is constant). In fact, let us make the ansatz $\mathbf{r}(t) \equiv \mathbf{r}_1(t) + \mathbf{r}_2(t)$, where \mathbf{r}_1 and \mathbf{r}_2 are constrained to satisfy

$$\frac{d\mathbf{r}_1(t)}{dt} = \mathbf{E}(\mathbf{r}_1) \times \mathbf{B}/B^2, \quad \frac{d^2\mathbf{r}_2(t)}{dt^2} = \frac{e}{m} \frac{d\mathbf{r}_2}{dt} \times \mathbf{B} \quad (5)$$

that is, the “guiding center” $\mathbf{r}_1(t)$ moves along equipotentials in the plane normal to \mathbf{B} , while $\mathbf{r}_2(t)$ performs cyclotron motion around the guiding center. It is clear that this ansatz satisfies Newton’s equation up to terms of order of the gradients of \mathbf{E} , so in the limit of sufficiently slow variation of the macroscopic potential it should be a good approximation.

Consider now a collection of electrons in a constant electric field \mathbf{E} and magnetic field \mathbf{B} . If we average over a time long compared to the cyclotron frequency ω_c , then independently of the phases of the different electrons the average electric current will be simply given by

$$\mathbf{J} = nev_0 = ne(\mathbf{E} \times \mathbf{B})/B^2 \quad (6)$$

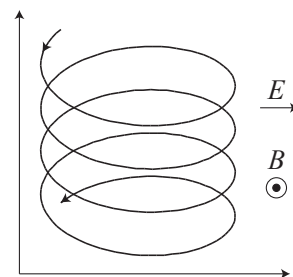
In the standard geometry, in which \mathbf{B} is normal to the plane containing the electrons, this means that the conductivity tensor is given by

$$\sigma_{xx} = \sigma_{yy} = 0, \quad \sigma_{xy} = -\sigma_{yx} = ne/B \quad (7)$$

or equivalently

$$\rho_{xx} = \rho_{yy} = 0, \quad \rho_{xy} = -\rho_{yx} = B/ne \quad (8)$$

It is interesting to obtain this result in a different way, which shows that it is much more general than indicated above. Consider a system of electrons moving in free space with arbitrary *translation-invariant* mutual interaction, subject to \mathbf{E} and \mathbf{B} and also to



collisions with static impurities which may be adequately described by a phenomenological collision time τ . Then, since the el-el interactions cannot change the electric current \mathbf{J} , the equation of motion of the latter is

$$\frac{d\mathbf{J}}{dt} = \frac{ne^2}{m} \mathbf{E} + \left(\frac{\mathbf{J}}{ne} \times \mathbf{B} \right) - \mathbf{J}/\tau \quad (9)$$

Setting the RHS of this equation to zero in a steady state, we obtain

$$\mathbf{E} = \rho_0 \mathbf{J} - \mathbf{J} \times \mathbf{B}/ne \quad (10)$$

where $\rho_0 \equiv m/ne^2\tau$ is the zero-magnetic-field resistivity. That is, the *resistivity* tensor has the simple form

$$\rho = \begin{pmatrix} \rho_0 & B/ne \\ -B/ne & \rho_0 \end{pmatrix} \quad (11)$$

and ρ_{xy} does not depend explicitly on the ratio $\omega_c\tau$. The *conductivity* tensor is a little more complicated:

$$\sigma = \frac{1}{1 + \omega_c^2\tau^2} \begin{pmatrix} \sigma_0 & (B/ne)\sigma_0^2 \\ (-B/ne)\sigma_0^2 & \sigma_0 \end{pmatrix}, \quad \sigma_0 \equiv \rho_0^{-1} \quad (12)$$

In the limit $\omega_c\tau \rightarrow \infty$ we have $\sigma_{xx} \rightarrow 0$, $\sigma_{xy} \rightarrow ne/B$, as we got from our earlier calculation.

In the real-life case which occur in experiments, the electrons are moving not in free space but in a crystalline lattice (and current is degraded by collisions with phonons as well as with static impurities). In the general case this gives rise to some subtle complications.⁵ However, in the cases of physical interest, when the Fermi wave vector is tiny on the scale of the first Brillouin zone, the above simple approach should be valid⁶ provided the true electron mass m is replaced by the effective mass m^* and the relaxation time includes the effects of phonons.

Quantum mechanics of a single electron in a magnetic field⁷

Consider a single electron moving freely in a plane (the xy -plane) under the influence of a magnetic field \mathbf{B} normal to the plane. Classically, this is the $\mathbf{E} = 0$ limit of the problem studied above, so the drift velocity is zero and the electron simply executes a periodic circular motion in the plane with the cyclotron frequency $\omega_c \equiv eB/m$ (or in the more general case eB/m^*). If we invoke the correspondence principle, we would infer that at least in the large-amplitude (semiclassical) limit quantum-mechanical effects would give rise to energy levels spaced by $h/\tau \equiv \hbar\omega_c$. Let's now see in detail how this comes about:

The free-electron Hamiltonian in the field $B\hat{z}$ has the simple form:

$$\hat{H} = \frac{1}{2}m(\dot{v}_x^2 + \dot{v}_y^2) \quad (13)$$

⁵Mostly connected with the possibility of Umklapp processes.

⁶At least in the limit $\omega_c\tau \gg 1$.

⁷Yoshioka, sections 2.2-3.

where \mathbf{v} is the (kinematic) velocity operator: this is related to the canonical momentum operator $\hat{\mathbf{p}}$ by

$$\hat{\mathbf{v}} \equiv m^{-1}(\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{r})) \quad (14)$$

where the electromagnetic vector potential $\mathbf{A}(\mathbf{r})$ is related to the magnetic field \mathbf{B} by $\mathbf{B} = \text{curl } \mathbf{A}$. The components of the velocity *fail to commute* for finite \mathbf{A} :

$$[\hat{v}_x, \hat{v}_y] = \frac{ie\hbar}{m^2} (\partial_x A_y - \partial_y A_x) \equiv \frac{ie\hbar}{m^2} B \quad (15)$$

Let us introduce, as well as the characteristic unit of time ω_c^{-1} , a characteristic length $l_M \equiv (\hbar/eB)^{1/2}$ whose significance will shortly become clear, and introduce a dimensionless velocity \mathbf{V} by

$$\mathbf{v} \equiv (l_M/\tau_c)\mathbf{V} \equiv (e\hbar B/m^2)^{1/2} \quad (16)$$

Then the commutation relations of the components of $\hat{\mathbf{V}}$ are

$$[\hat{V}_x, \hat{V}_y] = i \quad (17)$$

and the Hamiltonian has the form

$$\hat{H} = \frac{1}{2}\hbar\omega_c(\hat{V}_x^2 + \hat{V}_y^2) \quad (18)$$

The problem defined by eqns. (17) and (18) is of course nothing but that of the simple harmonic oscillator, and we can immediately write down the energy levels:

$$E_n = (n + \frac{1}{2})\hbar\omega_c \quad (19)$$

which of course perfectly satisfies the correspondence principle.

However, since the original density of states (before the magnetic field was turned on) was proportional to the area of the sample, and since the application of the magnetic field cannot change the average DOS if taken over a sufficiently large energy range, it follows that the energy levels we have found must be massively degenerate. Quantitatively, the original DOS (per spin state per valley) was $m/2\pi\hbar^2$ per unit area of surface. The new DOS is simply $1/\hbar\omega_c$, so it follows that the degeneracy per unit area must be

$$N = \hbar\omega_c(m/2\pi\hbar^2) \equiv eB/h \quad (20)$$

In other words, for any given level n there is *exactly one state per flux quantum h/e in the plane*. The different values of n are said to correspond to different *Landau levels*: for our purposes the most important is the lowest Landau level (LLL) corresponding to $n = 0$.

We now define a quantity which is central to the QHE, namely the *filling factor* ν . For simplicity I first consider the case of a single “valley” and a single spin population. Then the definition of ν is

$$\boxed{\nu \equiv \text{filling factor} \equiv \text{no. of electrons / flux quantum}} \quad (21)$$

We need to digress briefly at this point to discuss possible “spin” and “valley” quasi-degeneracies. Generally speaking, the Zeeman (spin) splitting E_Z will be small compared to the cyclotron energy $\hbar\omega_c$. At first sight this is surprising, since for a free electron the ratio is (close to) unity. However, if we consider for example GaAs, the g -factor is about 1/4 of the free electron value while the effective mass is only about 0.07 of that of a free electron; thus the ratio $E_Z/\hbar\omega_c$ is only about 0.015. As a result, if we could assume the electrons in GaAs to be noninteracting, then the sequence of filling of the levels (n, σ) (where $n = 0, 1, 2, \dots$ denotes the Landau level and $\sigma = \uparrow, \downarrow$ the spin projection on the magnetic field) would be as follows:

$$\begin{aligned}
 0 < \nu < 1 & \quad (0, \uparrow) \text{ partially filled, rest empty} \\
 1 < \nu < 2 & \quad (0, \uparrow) \text{ completely filled, } (0, \downarrow) \text{ partially filled, rest empty} \\
 2 < \nu < 3 & \quad (0, \uparrow), (0, \downarrow) \text{ completely filled, } (1, \uparrow) \text{ partially filled, rest empty} \\
 & \quad \text{etc.}
 \end{aligned} \tag{22}$$

However, we cannot necessarily assume that in the presence of the Coulomb interaction this filling scheme is maintained (in so far as it makes sense to still talk of a “filling scheme”); for example, we cannot exclude a priori a partial filling of $(0, \downarrow)$ even for $\nu < 1$.

In systems such as Si where there is more than one “valley” (conduction-band minimum) we need to take this also into account. However, there is an important difference with the spin degree of freedom, since while surface effects are apparently sufficient to reduce the six valleys of bulk Si to two, these two remain unsplit, and thus even with the neglect of the Coulomb interaction the filling sequence is ambiguous. This consideration is likely to matter most for the FQHE; however, fortunately the vast majority of experiments on the latter have been done on GaAs heterostructures, where this complication is absent (bulk GaAs has only one valley).

We will adopt the standard definition (21) of the filling factor ν *independently* of the number of spin species and valleys. Then with this definition the experimental finding is that the IQHE occurs around integral values of ν and the FQHE around odd-denominator fractional values (except for $\nu = 5/2$); in each case the Hall resistance is $h/\nu e^2$. Note that by contrast the traditional magnetic-quantization effects in 3D metals (dHvA, etc.) correspond to values of the filling factor of order $\epsilon_F/\hbar\omega_c \gtrsim 10^4$.

Representations of Landau-level wave functions

For the simple case considered so far (electrons moving freely in plane in constant magnetic field) the massive degeneracy of the wave functions corresponding to a given level implies that many different representations are possible for the wave functions. However, if the degeneracy is split (e.g. by a spatially varying electrostatic potential) then much of this freedom is lost. Irrespective of this, and whatever the form of the potential, we always have some freedom in choosing the gauge of the magnetic vector potential (recall that given any vector potential $\mathbf{A}(\mathbf{r})$ which satisfies $\mathbf{B} = \text{curl } \mathbf{A}$, we can always make the substitution $\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi$, where χ is an arbitrary single-valued scalar function,

without changing the magnetic field \mathbf{B}). Generally speaking, for any given choice of scalar potential there exists a “natural” choice of gauge which makes the calculations of the eigenfunctions easiest (though it is easy to find only in a few simple cases such as those to be discussed).

Common to all representations is a particular characteristic length, which we have already met, namely the *magnetic length*

$$l_M \equiv (\hbar/eB)^{1/2} \quad (23)$$

Note that (a) this quantity is completely independent of all materials parameters (b) the area corresponding to a single flux quantum is $2\pi l_M^2$ (c) the numerical value of l_M corresponding to a field of 1 T is approximately 250 Å.

One obvious representation is associated with the gauge choice $\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathbf{r} \times \mathbf{B}$. Then, in the dimensionless units, we have

$$\begin{aligned} V_x &\equiv \partial_x - \frac{1}{2}y \\ V_y &\equiv \partial_y + \frac{1}{2}x \end{aligned} \quad (24)$$

and the Schrödinger equation becomes

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{4}(x^2 + y^2) - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)\psi = E\psi \quad (25)$$

Suppose that we assume no dependence on the polar angle ϕ ; then the last term on the LHS vanishes and we have a simple 2D harmonic oscillator. The groundstate wave function is given by

$$\psi_0(r) = \frac{\sqrt{2\pi}}{l} \exp -r^2/4l^2 \quad (26)$$

and higher LL's are given in terms of the usual Hermite polynomials. However, if we make this choice the origin can at first sight be anywhere. We know however that the degeneracy must be 1/flux quantum, so the “allowed” choices of origin are spaced by $\sim \sqrt{2\pi}l$. This choice may be actually convenient for potentials having a complicated spatial variation.

A second representation is naturally associated with the so-called Landau gauge

$$A_x = -yB, \quad A_y = A_z = 0 \quad (27)$$

The states corresponding to the n -th Landau level are given by

$$\psi_n(x, y) = \exp(ikx) \cdot \phi_n(y - kl^2) \quad (28)$$

where $\phi_n(x)$ is the n -th linear oscillator state. Since for a finite slab the wave function must satisfy periodic boundary conditions at the edges, k can take only the values $2\pi p/L_x$ where p is integral. Also, if the states are confined in the y -direction with length L_y , k

can (from (28)) take values only between 0 and L_y/l^2 . This gives a total of $L_x L_y / 2\pi l^2$ states per Landau level which is right.

This representation is very convenient for considering the effect of a constant dc electric field \mathcal{E} . Suppose the field is in the y -direction. Then the Hamiltonian is still invariant against translation in the x -direction, so we can take out a factor $\exp(ikx)$ as before and write

$$\psi_n(x, y) = \exp(ikx) \cdot f(y) \quad (29)$$

where $f(y)$ obeys the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{2m} (\hbar k - eBy)^2 - e\mathcal{E}y \right) f(y) = Ef(y) \quad (30)$$

(E is the total eigenvalue, not that associated only with the y -motion).

We define a “guiding center” coordinate y_0 by

$$y_0(k) = -kl_M^2 + e\mathcal{E}ml_M^4/\hbar^2 \quad (31)$$

and thus obtain

$$\left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + m\omega_c^2(y - y_0)^2 \right\} f(y) = \left\{ E + e\mathcal{E}y_0 - \frac{m}{2} \left(\frac{\mathcal{E}}{B} \right)^2 \right\} f(y) \quad (32)$$

The solution is evidently a SHO eigenfunction $H_n(y)e^{-y^2/4l_M^2}$ with energy

$$E(n, y_0) = (n + \frac{1}{2})\hbar\omega_c - e\mathcal{E}y_0 + \frac{m}{2} \left(\frac{\mathcal{E}}{B} \right)^2 \quad (33)$$

The interpretation of the three terms is that the first term is the KE of cyclotron motion, the second is the PE of the guiding center in the electric field potential $-e\mathcal{E}y$ and the third is the KE associated with the classical drift velocity $|v_0| = \mathcal{E}/B$, which is in the x -direction: to see that the QM expectation value of v_0 is the same as the classical value, write

$$v_0 = \frac{1}{m} \langle p_x + eyB \rangle = \frac{1}{m} (\hbar k + eyB) \quad (34)$$

Using the relation (above) between y_0 and k and the definition of l_M^2 , this becomes

$$v_0 = \omega_c \langle y - y_0 \rangle + el_M^2 \mathcal{E} / \hbar^2 = \mathcal{E} / B \quad (35)$$

so $j = (e\mathcal{E}/B)/L$ (since the expectation value of $\langle x \rangle$ vanishes for any SHO eigenstate $\psi_n(x)$). Note that the above solution is *exact*⁸, independently of the relative magnitude of \mathcal{E} and B . Also note that the spacing of y_0 and hence the DOS is independent of \mathcal{E} .

It is intuitively plausible that the states we have constructed can be used to find an approximate solution to the problem of quantum motion in a magnetic field and an arbitrary strong, but slowly varying potential. In fact, it is fairly clear that provided that the electrostatic potential (or more generally any kind of external potential, which

⁸In the nonrelativistic limit. Clearly if we were to take the limit $B \rightarrow 0$ at finite \mathcal{E} , the drift velocity predicted by either the classical or quantum calculation $\rightarrow \infty$ and eventually we have to worry about relativistic corrections.

should play the same role) is slowly varying on the scale of l_M , we can obtain a solution which looks locally like that above one, with \mathcal{E} replaced by the local value of $-\nabla V$. The locus of the “guiding center” of such a state should be a contour of constant V , and the current carried by an electron in it should be given by

$$\mathbf{j} = e\mathbf{v} = -\frac{e}{B}(\hat{\mathbf{z}} \times \nabla V) \quad (36)$$

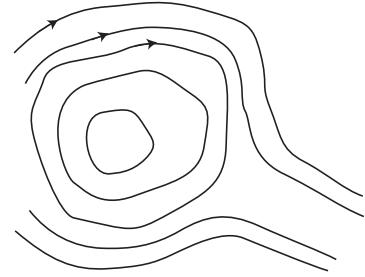
Notice that this current is divergence-free, as must be the case for an energy eigenstate:

$$\text{div } \mathbf{j} \sim \text{div}(\hat{\mathbf{z}} \times \nabla V) \equiv 0 \quad (37)$$

We should expect intuitively (and it can be confirmed from a more detailed analysis) that the average area corresponding to a single state is the area of a single flux quantum. Note however that depending on the detailed geometry of the potential, some states may be localized in the neighborhood of a potential minimum or maximum (circulating clockwise around a minimum and anticlockwise around a maximum), while others may be extended over the whole system.

We need still to consider one more representation, the so-called circular or symmetric one. This is most naturally associated with the radial gauge, which in plane polar coordinates has the form $A_r = 0$, $A_\theta = \frac{1}{2}rB$. It is possible to write down the exact form of (a possible set of) degenerate eigenfunctions (cf. above): in fact, the $n = 0$ (LLL) have the form, when expressed in terms of $z = x + iy$,

$$\psi_{0,l}(r, \theta) \equiv \psi_{0,l}(z) = \text{const. } z^l \exp -|z|^2/4l_M^2 \quad (38)$$



However, it is actually more instructive to examine the approximate form of the wave functions for large l . In terms of polar coordinates r, θ we have

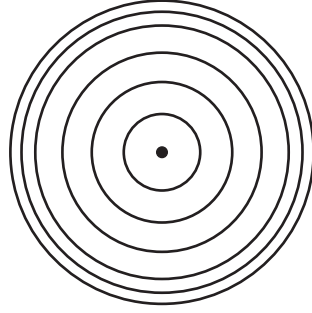
$$-\frac{\hbar^2}{2m} \left\{ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\partial}{\partial \theta} - \frac{ie}{\hbar} r^2 B \right)^2 \right\} \psi(r, \theta) = E\psi(r, \theta) \quad (39)$$

It is clear that it is possible to choose $\psi(r, \theta)$ to be of the form $\exp il\theta R_l(r)$ where to preserve single-valuedness l must be integral: for a reason which will become clear we also choose it to be positive. Then the radial wave function has the form

$$-\frac{\hbar^2}{2m} \left\{ \frac{1}{r} \frac{d}{dr} r \frac{dR_l}{dr} + \frac{1}{r^2} (l - r^2/2l_M^2)^2 \right\} R_l(r) = ER_l(r) \quad (40)$$

It is intuitively clear that for large l the wave function R_l will be confined to a region close to the value r_l of r given by

$$r_l \equiv (2l l_M^2)^{1/2} \quad (41)$$



Notice that a circular path with this radius encloses an area of $2\pi l l_M^2$, i.e. exactly l quanta of flux. For large l we can approximate the derivative term by $d^2 R_l / dr^2$, and expand the “potential” term in powers of $r - r_l$ up to second order: in this way we obtain

$$\left\{ -\frac{\hbar^2}{2m} \frac{d^2 R_l}{dr^2} + \frac{1}{2} m \omega_c^2 (r - r_l)^2 \right\} R_l = E R_l \quad (42)$$

This equation, when expressed in terms of the variable $x \equiv r - r_l$, is independent of l and is exactly of the form of the TISE of a SHO with frequency ω_c . Hence the solutions are the usual Hermite polynomials $H_n(r - r_l)$ with energies $E_n = (n + 1/2)\hbar\omega_c$. For the LLL, ($n = 0$), the extent of the wave function in the radial direction is $\sim \sqrt{\hbar/m\omega_c} \sim l_M$. Since the spacing between rings is $\sim l^{-1/2} l_M$, each radial wave function overlaps $\sim l^{1/2}$ of its neighbors (orthogonality is automatically guaranteed, within a given LL, by the angular function). Note that the approximation is not valid for very small values of l .

With a view to a thought-experiment to be considered in the next lecture, we need a slight generalization of the above argument. Consider a “Corbino-disk” geometry, that is, the above disk with a circular hole in its center through which we can apply an AB flux Φ . The generalization of eqn. (39) to this geometry is

$$-\frac{\hbar^2}{2m} \left\{ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\partial}{\partial \theta} - \frac{i\Phi}{\Phi_0} - \frac{ie}{2} \frac{r^2}{\hbar} B \right)^2 \right\} \psi(r, \theta) = E \psi(r, \theta) \quad (43)$$

The l -th state still encloses exactly l quanta of flux; the only difference is that now we have

$$r_l = (2(l - \Phi/\Phi_0) l_M^2)^{1/2} \quad (44)$$

With this choice of r_l , eqn. (42) is unchanged.