The quantum Hall effect: miscellaneous topics.

The $\nu = 1/2$ state of quantum Hall systems.

Although the fractional quantum Hall effect observed in at odd-denominator fractions is not always perfect, (the diagonal resistivity ρ_{xx} does not always sink to zero, and the plateaux are not necessarily completely flat), one can say that at least when the denominator is not two large (say $\nu \leqslant 11$) there is almost always, in the best samples, a clearly visible signature in both ρ_{xx} and the Hall conductance Σ_{xy}. By contrast, for most even-denominator fractions and in particular at $\nu = 1/2$ no effect is seen in either quantity (cf. e.g. Yoshioka fig. 1.11): the Hall conductance in this region is close to the straight line $\Sigma_{xy} = \nu e^2/h$ which would be predicted by a naive classical theory, and the diagonal resistivity is flat with at most a very shallow dip (not at all comparable to the much more pronounced dips seen e.g. at $\nu = 5/11$ and $6/11$, ibid. fig. 7.1). That something interesting may be going on at and around $\nu = 1/2$ was nevertheless originally indicated by the behavior of the velocity and attenuation of acoustic waves,[1] which show a pronounced dip and peak respectively in the neighborhood of $\nu = 1/2$, well separated from the $\nu = 5/11$ and $6/11$ fractions.

What might one expect to happen around $\nu = 1/2$ in the composite-fermion picture? Recall that in this picture the system of real electrons in a field B is visualized in terms of a system of composite fermions with filling fractions ν^* (each composite fermion being composed of a real fermion plus a fictitious flux $2p\phi_0$ where p is integral), subject to a fictitious magnetic field B^*. The relation between the actual and composite-fermion pictures is

$$\nu = \nu^*/(2p\nu^* \pm 1) \tag{1}$$

$$B = \pm B^* + 2pn_s\phi_0 \tag{2}$$

The sign has to be chosen so as to make B always positive. Inverting these equations and remembering that $\nu \equiv n_s\phi_0/B$, we have

$$\nu^* = \pm\frac{\nu}{1 - 2p\nu} \tag{3}$$

$$B^* = \pm(1 - 2p\nu)B \tag{4}$$

Suppose now that we put $\nu = \frac{1}{2} + \epsilon$, where the quantity ϵ is "small" but can have either sign. Then with $p = 1$ eqns. (3, 4) give

$$B^* = \epsilon B \tag{5}$$

$$\nu^* = 1/(2|\epsilon|) + \mathcal{O}(1) \tag{6}$$

In other words, the equivalent composite-fermion system is sitting in a *very small* magnetic field and thus has a very large value of its effective filling fraction. We can obtain

[1] The waves were propagated on the (outer) surface of an AlGaAs/GaAs heterojunction; because these materials are piezoacoustic, an electric field is thereby generated on the electrons in the (internal) inversion layer.

the effective number per unit area of composite fermions n_s^* using definition $\nu^* = n_s^* \phi_0 / B$ and the relation (from (3-4) $\nu^* B^* = \nu B$; the result is that n_s^* is simply equal to n_s, the original number of electrons per unit area. (This relation of course holds for arbitrary ν viewed in the composite-fermion picture, and indeed is a vital component in this picture). What about the charge of the composite fermions? Since they simply consist of an electron plus two fictitious "flux quanta", this is clearly just the electron charge e.

Thus, at exactly $\nu = 1/2$, the composite-fermion picture allows us to regard the real system of electrons as equivalent (within the mean-field approximation) to a gas of fermions with the same areal density and charge as the actual system, in *zero magnetic field*! But we know a good deal about this latter system; in particular, if we assume that the electron-electron interactions do not lead to crystallization, superconductivity or any other kind of ordered state, then at low temperatures the system is just a *textbook Fermi liquid*, with a Fermi wave vector k_F which, since we are dealing with a polarized system (only one Zeeman component populated) is exactly $\sqrt{2}$ times what it would be for the "textbook" (unpolarized) system of the same density. Moreover, for small ϵ, i.e. small deviations of the filling factor ν from 1/2, the composite-fermion system should experience an effective magnetic field $\mathbf{B}^* = \epsilon \mathbf{B}$ which may be of either sign (depending on the sign of ϵ) relative to the physical field $\mathbf{B}$, but is in any event small; in particular, even though the ration $\hbar\omega_c/\epsilon_\mathrm{F}$ may be quite comparable to unity, for small enough ϵ the corresponding ratio for the composite fermion, $\hbar\omega_c^*/\epsilon_\mathrm{F}$, can be made arbitrarily small compared to 1, so that we are in the regime where (were we in 3D rather than 2D) the standard dHvA effect and related effects would be seen.

Can we test this remarkable prediction? Perhaps the most obvious test would be to verify that the low-temperature specific heat has the Fermi-liquid form *const.*T, where the constant is proportional to the effective mass m^*. However, this is not very practical, since in view of the 2D nature of the system the specific heat is very difficult if not impossible to measure (and in any case, as we shall see below, the behavior of m^* in the limit $\nu \to 1/2$ is problematic). A more promising approach is to try to detect the effects of the orbital motion which the (charged) composite fermions should execute in the weak effective field B^*. According to the standard semiclassical argument, in such a field the fermions should describe circular orbit whose radius R^* is given by

$$R_c^* = m^* v_\mathrm{F}/eB^* = \hbar k_\mathrm{F}^*/eB^* \tag{7}$$

which from the relations $n_s = k_\mathrm{F}^2/4\pi$, $\nu \equiv n_s\phi_0/B$ can be written in the simpler form

$$R_c^* = (\epsilon^{-1} + 2) k_\mathrm{F}^{*-1} \approx \epsilon^{-1} k_\mathrm{F}^{*-1} \tag{8}$$

To check that the quantity R_c^* has some physical significance in the real electron system, various experiments can be used. In one type one uses an "antidot superlattice" and compares the pattern of the Hall effect with that in the absence of the superlattice; the behavior around $B = 0$ and $B = 12\,\mathrm{T}$ (the field for which $\nu = 1/2$) is similar,[2] see

[2] when scaled by a factor of $\sqrt{2}$ to allow for the fact that close to B (not B^*) $= 0$ the electron system is unpolarized.

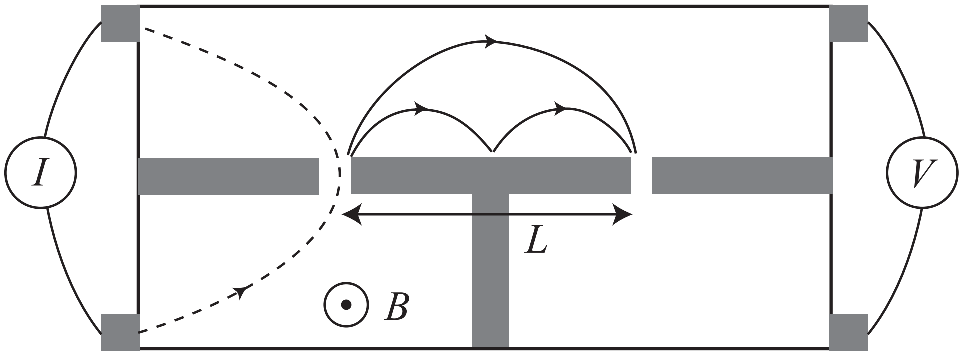

Figure 1:

Yoshioka fig. 7.3. A second set of experiments looks for evidence of resonance behavior when $2R_c^*$ becomes equal to some other characteristic length such as the sample width or the wavelength of an imposed sound wave ("Weiss oscillations").

However, possibly the most spectacular experiments are those on "magnetic focusing".[3] The experiment is done on a high-mobility GaAs-AlGaAs heterostructure with $n_s \sim 10^{11}\,\mathrm{cm}^{-2}$; what is measured is $R_{xx} \equiv V/I$ (see figure). The sense of the current flow (dashed line) and magnetic field is such that electrons are deflected, in the *real* field, to the right. The authors first measured R_{xx} in *small* fields B, where no quantum Hall effect should occur, and found that the pattern shows characteristic anomalies whenever the condition

$$R_c = \hbar k_\mathrm{F}/eB = L/n \tag{9}$$

is met, i.e. the pattern is periodic in B with periodicity

$$\Delta B = \hbar k_\mathrm{F}/eL \quad (= 25\,\mathrm{mT}) \tag{10}$$

This is of course not unexpected (it had been seen in earlier experiments). The interesting result is what is found for B close to 8 T (corresponding to $\nu = 1/2$). On the *upper* side ($\nu > 1/2$) the pattern is very similar to the low-field one, but with a spacing

$$\Delta B^* = 36\,\mathrm{mT} \tag{11}$$

Since 36 mT is close to $\sqrt{2}$ times 25 mT, this is very close to the periodicity ($R_c^*(B) = L/n$) one would expect for the composite fermions (which, remember, have $k_\mathrm{F}^* = \sqrt{2}k_\mathrm{F}$). On also finds that on the *low* side of $B = 8\,\mathrm{T}$ (i.e. for ν somewhat smaller 1/2) there is no such correspondence; this is exactly what one would expect on the composite fermion picture, since for such conditions B^* is negative and the composite fermions passing through the left aperture would be deflected to the left and not affect the system in the right half.

[3] Goldman et al., PRL **72**, 2065 (1994). Note that Yoshioka's fig. 7.4 apparently misidentifies the position of the voltage leads.

A final set of experiments related to the composite fermion picture at $\nu = 1/2$ are the original ones on ultrasonic absorption. As noted above, this is believed to be effectively due to the absorption by the 2D electron gas of the electric field generated in the GaAs via the piezoelectric effect, and is thus a measure of the longitudinal conductivity $\sigma_{xx}(q)$ of the 2DEG, where q is the wave vector of the absorbed phonon (which is presumably $\ll k_\mathrm{F}$). Now we have quite generally

$$\sigma_{xx}(q) = \frac{\rho_{yy}(q)}{\rho_{xy}^2(q) + \rho_{xx}(q)\rho_{yy}(q)} \tag{12}$$

In the system of an (integral or fractional) QHE the RHS of eqn. (12) is zero, while in the transition regimes it is nonzero; in fact of order e^2/h. Thus one expects that the transmitted ultrasound amplitude (proportional to the inverse of the absorption, hence (approximately) of $\sigma_{xx}(a)$) should peak at the QHE plateaux and this is indeed seen in the experiments. On the other hand, it is a known result that for a free 2DEG in a magnetic field in the region $ql \gg 1$ (hence, usually, $\omega\tau \ll 1$) the longitudinal resistivity should be given by the simple formula

$$\rho_{yy}(q) = \frac{h}{e^2}\frac{q}{k_\mathrm{F}} \tag{13}$$

In attempting to apply these results to the composite fermion picture of the $\nu = 1/2$ state one runs up against an ambiguity: should we apply eqn. (12) to the conductivity components of the real electron system, or to the fictitious composite fermion system? The problem is that in the latter case B^* is close to zero, so the ρ_{xy}^2 term in the denominator of the RHS of (12) is small and we have effectively $\sigma_{xx}(q) \sim (k_\mathrm{F}/q)\,(e^2/h)$, leading to a strong absorption (drop in the transmitted amplitude) around $\nu = 1/2$; this actually seems to be consistent with the experiments (see Yoshioka fig. 7.1). If on the other hand with Yoshioka we apply (12) to the real electron system, then since for this $\sigma_{xx} \sim e^2/h$ we find $\sigma_{xx}(q) \sim (e^2/h)\,(q/k_\mathrm{F})$, i.e. the transmitted amplitude should *increase*, though not as much as at the IQHE and FQHE peaks. This problem appears to be currently unresolved...

Effects of spin, valley and layer degeneracy.[4]

Up to now, we have been implicitly assuming that the only relevant quantum numbers of the electrons which participate in the QHE are those related to their in-plane orbital motion (namely the Landau-level number n and the orbital angular momentum l). However, this is not necessarily true. First, whatever the material and geometry, the electrons always posses a spin degree of freedom, which interacts with the applied magnetic field (irrespective of the direction of the latter) by the Zeeman term. Secondly, in Si MOSFET's (though not in GaAs heterostructures) two of the original six originally degenerate "valleys" of the bulk crystal remain degenerate even in the presence of a surface (which removes the degeneracy with the other four). Thirdly, in recent years

[4]General reference: Chakraborty and Pietilainen, The Quantum Hall Effect, 1988, ch. 5

it has become possible to construct *bilayer* QH systems, with two effectively 2D layers separated by an insulating layer; in this case (as in the case of the two valleys in Si) we can represent the extra degree of freedom by a "pseudospin" τ such that $\tau_z = \pm 1$ corresponds to an electron localized in one layer (valley) or the other. Up to now we have implicitly assumed that, apart possibly from filled and hence presumably inert Landau levels, all the relevant electron states correspond to a single value of σ_z and, where relevant, τ_z.

To assess the possible effects of the spin and valley degrees of freedom we need to estimate some orders of magnitude. The valley degeneracy in Si is usually unsplit,[5] but fortunately does not exist in the GaAs heterostructures which are nowadays the almost universally favored systems for experiments on the QHE, so I shall neglect it from now on (if present, it can be handled similarly with the spin degree of freedom). As to the Zeeman energy, E_Z we have already seen that for GaAs in a *perpendicular* magnetic field it is about 1/70 of the cyclotron energy, E_{cyc}. However, the cyclotron energy is proportional only to the component of the field perpendicular to the surface, while the Zeeman energy involves all components; hence the ratio is given in the general case by

$$\frac{E_Z}{E_{\text{cyc}}} \approx \frac{0.014}{\cos\theta} \tag{14}$$

where θ is the angle made by the field with the surface normal.

In any case, the ration E_Z/E_{cyc} is not the only relevant one; we also have to consider the Coulomb energy, whose general order of magnitude is $e^2/\epsilon\epsilon_0 l_M$; this quantity scales as $B^{1/2}$ and for GaAs at 1 T is about 4 meV (50 K). The cyclotron energy $E_{\text{cyc}} \equiv \hbar\omega_c = e\hbar B/m^*$ scales as B and at 1 T is about 2 meV (25 K) (so that the assumption $E_{\text{Coul}} \ll E_{\text{cyc}}$ often made in theoretical discussions is actually not well satisfied for most of the experimentally relevant fields). We see that whatever the orientation of the field, the Coulomb energy is large compared to the Zeeman energy at any field $\lesssim 10^4\,T$ (a currently quite unachievable value).[6] Consequently we cannot duck the question: Does the effect of the Coulomb interaction make it energetically favorable to depolarize the QH state, i.e. to allow the electrons to have both spin polarizations, depsite the fact that this costs some Zeeman energy?

To investigate this question it is useful to generalize the Laughlin wave function to the case of zero or partial spin polarization. This was done by Halperin as follows: let the coordinates of the up-spin electrons be labeled z_i and those of the down-spin electrons ξ_i. Then a possible ("Halperin") wave function is of the form, up to overall

[5] The spin-orbit interaction cannot split it even in the presence of magnetic field, since a term of the form $\mathbf{k}\cdot\boldsymbol{\sigma}$, while preserving invariance under time-reversal destroys invariance under spatial inversion.

[6] However, the above discussion, which is based entirely on order-of-magnitude estimates, ignores the fact that quantitative calculation of the energies of fractional quasiparticles (which to a first approximation are entirely Coulomb in origin) give values which are at least an order of magnitude small than $e^2/\epsilon\epsilon_0 l_M$ (there ought at least to be a 4π in the denominator!). Thus even at 15 T this energy may be smaller than the Zeeman term.

antisymmetrization

$$\Psi_{m_+,m_-,n}(z_1, z_2 \dots z_{N_\uparrow}; \xi_1, \xi_2 \dots \xi_{N_\downarrow}) = \prod_{i<j}^{N_\uparrow}(z_i - z_j)^{m_\uparrow} \prod_{i>j}^{N_\downarrow}(\xi_i - \xi_j)^{m_\downarrow} \prod_{i=1}^{N_\uparrow}\prod_{j=1}^{N_\downarrow}(z_i - \xi_j)^n \times \exp -\frac{1}{4}\left[\sum_{i=1}^{N_\uparrow}|z_i|^2 + \sum_{i=1}^{N_\downarrow}|\xi_i|^2\right] \tag{15}$$

where $N_\uparrow$, $N_\downarrow$ are respectively the total number of up-spin and down-spin electrons; thus $N \equiv N_\uparrow + N_\downarrow$ and the total spin S is $\frac{1}{2}(N_\uparrow - N_\downarrow)\hbar$. In order to guarantee the correct antisymmetry of the wave function under exchange of parallel-spin electrons, both $m_\uparrow$ and $m_\downarrow$ must be odd integers, but in the general case they need not be equal. Note also that even in the case of a completely unpolarized (spin singlet) state, where from symmetry one expects $m_\uparrow = m_\downarrow$, the exponent n may be different. Any given state of the form (15) is conventionally labeled by the exponents $m_\uparrow, m_\downarrow, n$; thus for example the state with $m_\uparrow = m_\downarrow = 3$, $n = 1$ (which as we shall see in a later lecture is a candidate for the $\nu = 5/2$ QH plateau) is called the "(3,3,1)" state.

Whether or not a partially polarized or unpolarized state of the form (15) is energetically competitive, at a given fractional[7] value of ν, with the simpler completely polarized states discussed in lectures 18 and 19, is a matter for detailed calculation. Since as usual in the theory of the QHE there is no "small parameter", one typically falls back on numerical computation for a small number of electrons. Such calculations tend to suggest that while the most robust FQHE states such as $\nu = 1/3$ are likely to be completely polarized, partial or no polarization may be favored for larger values of the denominator.

What about experiment? Direct measurement of the magnetization, e.g. by SQUID magnetometry, is difficult because of the small absolute number of spins involved. However, it is possible to measure the degree of spin polarization by an ingenious technique in which one photoexcites holes from the donors in the bulk GaAs, allows them to recombine with the 2D electrons and measures the degree of circular polarization of the recombination radiation. The analysis of the raw data is not entirely trivial, because the holes themselves tend to be polarized by the magnetic field, but this effect can be calibrated by using the data at $\nu = 1$, where 100% polarization of the 2DEG is expected. A series of experiments along these lines were performed by Kukushkin et al.[8] in 1997, with the following results: For all states with $\nu \leqslant 1$, full polarization was observed at low temperatures in fields above 4 T. However, at lower fields the $\nu = 2/3$, 2/5 and 3/5 states were observed to be only partially polarized. In all cases the degree of polarization decreases rapidly with increasing temperature. Some very interesting phenomena were observed at values of ν close to 1: while at $\nu = 1$ the state is fully polarized at all B,

[7] It seems virtually certain that the IQHE at even integral values of ν is unpolarized, and of course for $\nu > 2$ one is likely to get only partial polarization, because the LLL is likely to be filled for both spins.

[8] PRB **55**, 10607 (1997).

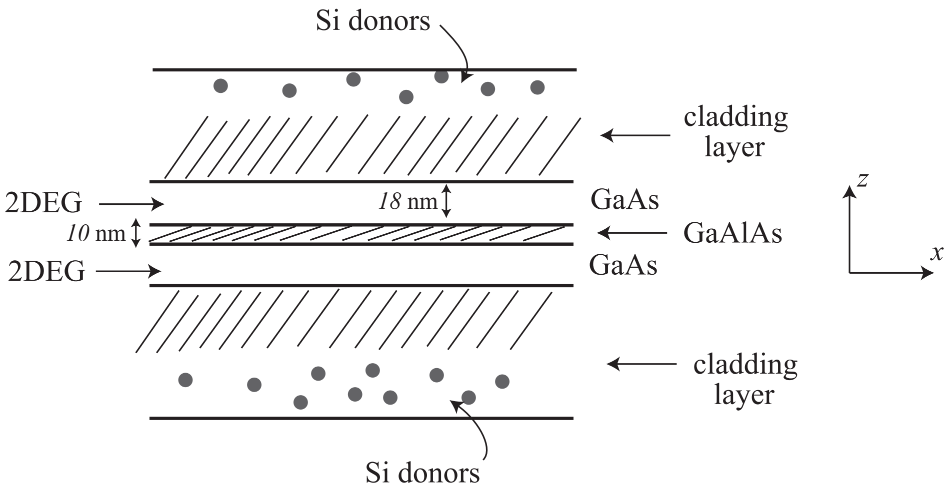

Figure 2:

for $B < 2\,\mathrm{T}$ there is considerable depolarization at close-by values. This is thought to reflect the presence of "skyrmions", a collective excitation involving the simultaneous excitation of a considerable number of spins. Agreement between theory and experiment on this appears to be qualitative but not entirely quantitative.

Another technique to measure the spin polarization of QHE systems is electrically detected NMR.[9] This has been used to study a phase transition which apparently occurs for the $\nu = 2/3$ state in low fields, between an unpolarized and a fully polarized state as a function of field.

Finally, let's turn to the quantum Hall effect in bilayer systems. These can be realized in double quantum well heterostructures: a typical experimental setup[10] is shown in fig. 2. The two 2D electron gases occupy the GaAs quantum wells; their areal density can be controlled individually by appropriate gating. However, the perpendicular magnetic field will normally be the same for both layers so that not only are the cyclotron and Zeeman frequencies identical but the ratio of filling factors ν_1/ν_2 is just the ration of the densities; it is possible to implement pretty much arbitrary values of this ratio. The normal notational convention is to denote the "total" filling factor $\nu_1 + \nu_2$ by ν_T.

There are clearly a substantial number of different energies relevant to this problem. In the first place, we have for each layer the "single-layer" energies $\hbar\omega_c$ and E_Z, and the intralayer Coulomb energy $V(r) \equiv e^2/\epsilon\epsilon_0 r$; as previously we denote the quantity $V(l_M)$ by V_c and note that it is the same for both layers (but we must remember also that the "characteristic" Coulomb energy which actually enters excitation energies etc. is αV_c where $\alpha \ll 1$). We will assume unless otherwise stated that not only $\hbar\omega_c$ but E_Z are $\gg \alpha V_c$, so that the system is fully spin-polarized as regards "real" spin (as distinct from pseudospin, see below). However, we also have several energies which are specific to the

[9]See O. Stern et al., PRB **70**, 075318 (2004).
[10]Charmagne et al., PRB **78**, 205310 (2008).

bilayer problem:

(1) A possible (voltage-controlled) bias energy $\Delta\varepsilon$ between the two layers.

(2) A possible tunneling energy Δ between the two planes. In zero parallel magnetic field, Δ is simply a position-independent constant. However, by applying a field component $B_{\|}$ *parallel* to the planes we can make the phase of Δ depend on the in-plane position; e.g. if the field is along the y-axis

$$\Delta \to \Delta \exp i \int_{z_1}^{z_2} A(x)\, dz \equiv \Delta \exp i\phi(x) \tag{16}$$

The phase $\phi(x)$ is periodic (in a suitably chosen gauge) with a period L such that $LdB_{\|} = \phi_0$, where d is the effective interlayer distance (see below).

(3) The interplane Coulomb interaction: If we assume for simplicity that the dielectric constant of the "spacer" material is identical to that (ϵ) of the 2DEG layer (a fairly good approximation when the space is GaAlAs and the 2DEG GaAs), then in coordinate space the "raw" coupling is of course just the standard $e^2/4\pi\epsilon\epsilon_0 r_3$ where r_3 is the 3D distance. If we make the approximation that the thickness of the 2DEG layers is small compared to their separation, then we can write the interaction as

$$V_{12}(r) = e^2/4\pi\epsilon\epsilon_0\sqrt{r^2+d^2} \tag{17}$$

where d is the center-to-center layer spacing and r is the component of $\mathbf{r}_1 - \mathbf{r}_2$ parallel to the layers. It should be noted that although the expression (17) does not look that different from the in-plane interaction $V_{11} = V_{22} = e^2/4\pi\epsilon\epsilon_0 r$, the 2D Fourier transforms of the two quantities are quite different:

$$V_{11}(k) = V_{22}(k) = \text{const}\, k^{-1}, \quad V_{12}(k) = \text{const}\, k^{-1} \exp -kd \tag{18}$$

In real life, the thickness of the individual 2DEG layers is often not small compared to their separation, so one needs to define an "effective" interlayer distance d; the latter may be sensitive to the z-dependence of the wave function and thus adjustable, within limits, by variation of the gating voltage.

Let us start by assuming no tunneling between the two layers of the bilayer, and moreover assume for the moment that the interlayer bias $\Delta\varepsilon$ is zero. At first sight the problem then looks similar to that of "spinful" electrons on which the Zeeman field has been turned off, and indeed one can usefully introduce a "pseudospin" variable τ such that $\tau_z = \pm 1$ corresponding to the electron being on layer 1 and 2; in such notation the tunneling energy, were it important, would be represented by Δ times the xy-component of τ. However, even in the absence of tunneling and bias the bilayer problem is not in fact equivalent to that of a system of electrons with real spin in zero Zeeman field, because in contrast to that problem the Coulomb interaction depends on the relative pseudospin of the electrons involved: in fact

$$V_{\text{Coul}}(\boldsymbol{\tau}, \boldsymbol{\tau}'; r) = \frac{1}{2}\big[V_{11}(r) + V_{12}(r)\big] + \frac{1}{2}\big[V_{11}(r) - V_{12}(r)\big]\tau_{1z}\tau_{2z} \tag{19}$$

so in general V_{Coul} is not $SU(2)$ invariant in the $\boldsymbol{\tau}$-space.[11] It is intuitively clear that the qualitative features of the behavior are likely to depend strongly on the dimensionless ratio d/l_{M}. In the limit $d/l_{\text{M}} \to \infty$, we have two completely independent single-layer quantum Hall systems and they will behave appropriately. In the opposite limit $d/l_{\text{M}} \to 0$ (but still no tunneling allowed between the layers) the situation is genuinely equivalent to the case of a real-spin-1/2 system in zero Zeeman field, since now $V_{11}(r) = V_{12}(r)$ and the system prima facie has the full $SU(2)$ invariance in $\boldsymbol{\tau}$-space.

A particularly interesting case is when $\nu_T = 1$. For large values of d/l the capacitance effect due to the long-range part of the Coulomb interaction should enforce (for $\Delta\varepsilon = 0$) the "equipartition" result $\nu_1 = \nu_2 = 1/2$, and we know that at $\nu = 1/2$ the system can be regarded as a Fermi liquid of composite fermions; in particular, there is no plateau in R_{H}. In the opposite limit $d/l_{\text{M}} \to 0$, just as in the analogous real-spin case, it is generally believed that a good description of the system is the so-called Fertig wave function:

$$\Psi_{\text{F}} = \prod_{l \in \text{LLL}} \frac{1}{\sqrt{2}} (c_{1l}^\dagger + c_{2l}^\dagger)|\text{vac}\rangle \tag{20}$$

where $c_{jl}^\dagger$ creates an electron in the l-th state of the LLL in layer j. In other words, even though there is no tunneling between the layers, all electrons occupy the even-parity state $1/\sqrt{2}\,(|1\rangle + |2\rangle)$.

It is interesting to note that the Fertig state is closely related to the Halperin (111) state

$$\Psi_{111} = \prod_{j<k}^{N_1} (z_j - z_k) \prod_{r<s}^{N_2} (w_r - w_s) \prod_{j,r}^{N} (z_j - w_r) \exp -\frac{1}{4} \left[\sum_{j=1}^{N_1} |z_j|^2 + \sum_{r=1}^{N_2} |w_r|^2 \right] \tag{21}$$

In fact, the latter is just the projection of the Fertig state on to the manifold with given single-layer occupations N_1 and N_2 (see Jain, pp. 464-5).

In the limit of true $SU(2)$ invariance, the Fertig state (20) would be degenerate with states $(j = 1, 2)$

$$\prod_{l \in \text{LLL}} c_{jl}^\dagger |\text{vac}\rangle \tag{22}$$

In real life d/l_{M} is never zero, so there will be a capacitance term which disfavors this state (this is just the second term in (19), which clearly favors $\tau_{z\,\text{tot}} = 0$). However, we are still left (actually for any value of d/l_{M}) with a $U(1)$ invariance corresponding to rotation around the z-axis in τ-space, i.e. to the transformation

$$\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \to \frac{1}{\sqrt{2}}(|1\rangle + \exp i\phi |2\rangle) \tag{23}$$

which in the absence of tunneling is a good symmetry. Consequently, we got a situation which begins to look very like the standard *one*-plane XY-model used to describe e.g. the

[11] Jain eqn. (13.5) makes the second term proportional to $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$, which seems to be an error.

superfluidity of ^{4}He films (see lecture 8). Indeed, we expect that a spatial variation of the relative phase ϕ will cost an energy proportional to $(\nabla\phi)^2$, which can be written just as in the XY-model as

$$E_{\text{bend}} = \frac{1}{2}\rho_s \int (\nabla\phi)^2 \, d^2\mathbf{r} \tag{24}$$

From this analogy one would expect that in this limit ($d/l_{\text{M}} \to 0$) there is no true ODLRO at any nonzero temperature, but the system nevertheless shows the analog of "superfluidity" below a Kosterlitz-Thouless transition temperature given by

$$T_{\text{KT}} = \frac{2}{\pi}\rho_s(T_{\text{KT}}) \tag{25}$$

How would this "superfluidity" be manifested? A variation of the *relative* phase ϕ in the two layers corresponds to no electrical current, but it does correspond to a *counterflow*, which can be induced by contacting the two layers of the bilayer individually. An experiment along these lines was done by Kellogg et al.[12] on a bilayer system with $d/l_{\text{M}} = 1.58$, with the result shown in Jain fig. 13.6: when the *total* current in the two layers and the (common) voltage across them are measured, the pattern is essentially that of the standard IQHE with $\nu \to \nu_T$, and in particular at $\nu_T = 1$ there is fairly well-defined plateau in the Hall resistance R_{xy} and a dip in R_{xx} (though the corresponding features at $\nu_T = 2$ are much more "ideal"). By contrast, if one measures in the "counterflow" geometry there is no (finite-height) plateau at $\nu_T = 1$, in fact the Hall resistance tends to *zero*, while the longitudinal resistance appears to remain finite (though it does have a dip). This is strongly reminiscent of superfluidity; however, the true analog of this would be if R_{xx}, not $R_{xy} \to 0$, and there is no experimental evidence for this even for $T <$ calculated T_{KT}.

In view of the qualitatively quite different behavior of the system at $\nu_T = 1$ in the two limits $d/l_{\text{M}} \to 0$ and $d/l_{\text{M}} \to \infty$, the question arises whether there is a phase transition for intermediate values of d/l_{M}, or alternatively as a function of the charge-density imbalance $\Delta\nu \equiv \nu_1 - \nu_2$ (since for $\Delta\nu = 1$ the system is clearly confined to layer 1 and thus in the simple IQHE state). The most systematic investigation of this question is reported in a recent paper by Champagne et al.[13] who indeed see, via interlayer transport measurements, a transition occurring as a function of $\Delta\nu$, d/l_{M} and T. What is quite surprising about their results is that the "phase-coherent excitionic state" (i.e. a generalization of the Fertig wave function to the state $a|1\rangle + b|2\rangle$, where $|a|^2 = \nu_1$, $|b|^2 = \nu_2$) persists at least up to an imbalance $\Delta\nu$ of 0.5 (i.e. $\nu_1 = 3/4$, $\nu_2 = 1/4$).

Finally, yet more possibilities are opened up once we allow for a non-negligible interlayer tunneling rate (though this will clearly tend, by fixing the "optimal" value of ϕ, to inhibit the "excitonic superfluidity" described above). There is no space to discuss this topic here: see e.g. Yoshioka section 6.5.2.

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[12] PRL **93**, 036801 (2004).
[13] PRB **78**, 205310 (2008).