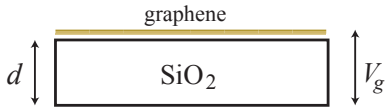
Graphene II: quantum Hall effect.

To introduce the subject of the QHE in graphene, let's start by considering the doping process. In the original experiments of Geim et al., the graphene layer was placed on top of a $\sim 300\,\mathrm{nm}$ thick SiO_2 slab and a gate voltage applied to it relative to the back face of the slab. Since the capacitance of the condenser so formed is $\epsilon\epsilon_0 A/d$ where d is the thickness and ϵ the dielectric constant of SiO_2, this gives for surplus electron (hole) number density per unit area

$$n_s = CV_g/e = (\epsilon\epsilon_0/de)V_g \approx 7.2 \times 10^{10}\,\mathrm{cm}^{-2} \tag{1}$$



Notice that V_g and hence n_s can have either sign (negative n_s means hole doping). It is possible to check eqn. (1) by measurements of the Hall effect in the "unquantized" regime, which is well attained at room temperature (cf. below), and it seems to be valid; that is, the Hall resistance R_H is given by $1/n_s e$ with n_s given by eqn. (1).[1] It is possible to apply voltages up to $\sim 100\,\mathrm{V}$ without breakdown of the SiO_2 and thus to obtain areal densities up to $\sim 10^{13}\,\mathrm{cm}^{-2}$, one or two orders of magnitude higher than what is usually possible in Si MOSFET's or GaAs heterostructures. Mobilities μ of up to $\sim 15000\,\mathrm{cm}^2/\mathrm{V\,sec}$ are routinely obtained; since they are almost temperature-independent, the belief is that they are due to impurity scattering and thus it may be possible to improve μ by around an order of magnitude. In fact, in a very recent experiment on a suspended graphene sheet, the mobility was $2.6\times10^5\,\mathrm{cm}^2/\mathrm{V\,sec}$, corresponding to a mean free path of $\sim 0.35\,\mu$ (comparable to the sample dimension).

Because of the unusual energy spectrum of the carriers in graphene, it is necessary to reconsider several of the standard formulae used in discussing the QHE, and in particular the dependence of various characteristic energies on n_s, B etc. In a traditional QHE system, the characteristic constant describing the spectrum (m^*) has the units of mass, while in graphene the spectrum is parametrized by the Fermi *velocity*. As we have seen, the DoS close to the Dirac points is linear in energy: taking into account the spin and valley degeneracy, we have

$$n_s = q_\mathrm{F}^2/\pi \tag{2}$$

and hence the Fermi energy is given by

$$\epsilon_\mathrm{F} \equiv \hbar v_\mathrm{F} q_\mathrm{F} = \hbar v_\mathrm{F}(\pi n_s)^{1/2} \tag{3}$$

For $n_s = 10^{13}\,\mathrm{cm}^{-2}$ ϵ_F is $\sim 4000\,K$, meaning that most of the samples studied are highly degenerate at room temperature. Note that for very small dopings ϵ_F can exceed the bias energy eV_g, so that further analysis of the doping process may be necessary in this regime.

Next, let's consider the mean free path. Just as in the standard case, the conductivity σ and mobility μ are related by $\sigma = n_s e\mu$ (from which we may obtain the experimental

[1] In the original experiment an "offset" voltage $V_0 \sim 50\,\mathrm{V}$ was seen, i.e. $n_s \propto (V_g - V_0)$ rather than $\propto V_g$; also the relation was not linear for small n_s. These effects have been removed in subsequent experiments.

value of μ) and we may write

$$\mu = e\tau/m_{\rm F}^* \tag{4}$$

where τ is the (elastic) collision time and $m_{\rm F}^*$ the effective mass at the Fermi energy. However, unlike in the standard case, this latter quantity is itself strongly filling-dependent, so it is easier to write μ in terms of the elastic mean free path l:

$$\mu = el/m_{\rm F}^* v_{\rm F} \equiv el/\hbar k_{\rm F} = (e/\hbar)l/(\pi n_s)^{1/2} \tag{5}$$

The conductivity (in zero magnetic field) is therefore given by

$$\sigma = \frac{e^2}{h} l(\pi n_s)^{1/2} \equiv \frac{e^2}{h}(k_{\rm F} l) \tag{6}$$

Consider next the energies associated with an external magnetic field. The g-factor of the π-electrons in graphene is very close to 2, so the Zeeman energy should be given by the usual expression $2\mu_{\rm B}B = (e\hbar/m)B$ where m is the bare electron mass and B the total field (independent of orientation). By contrast, the orbital (cyclotron) mass is strongly energy-dependent: if we evaluate it at the Fermi energy we have for the cyclotron energy

$$\hbar\omega_c \equiv e\hbar B/m^* = ev_{\rm F}B/k_{\rm F} \propto n_s^{-1/2} \tag{7}$$

(where B is now the field component normal to the plane). At $n_s \sim 10^{12}\,{\rm cm}^{-2}$ the cyclotron mass is about 0.02 of the bare mass, so similarly to GaAs heterostructures the cyclotron energy is a factor of order 50 times the Zeeman term; we therefore expect the considerations about the order of filling of Landau levels to be qualitatively similar (with one proviso discussed below).

Finally we consider the Coulomb energy. For a free graphene sheet the situation is straightforward: since the system is truly 2D, and in addition the polarizability of the C core levels is small, the interaction between the π^*-band conduction electrons (or the π-band holes) should have more or less the free-space Coulomb form $e^2/4\pi\epsilon_0 r$. For the case of epitaxial graphene things are somewhat less clear, since even if the substrate has a large dielectric constant ϵ it occupies only a half-space; in the literature this complications seems often to be ignored and one writes the Coulomb interaction as $e^2/4\pi\epsilon\epsilon_0 r$. If we put in numbers, then the "characteristic" Coulomb energy (the Coulomb interaction evaluated at the magnetic length $l_{\rm M} \equiv (\hbar/eB)^{1/2}$ is about 50 meV (600 K) at 1 T for freely suspended graphene, and of order ϵ^{-1} times this for the epitaxial configuration. Note that at a given value of filling fraction ν the ratio of V_c to $\hbar\omega_c$ is independent of the areal density:

$$V_c/\hbar\omega_c = \frac{1}{2}\nu^{1/2}\alpha_g \quad \alpha_g \equiv \frac{e^2}{4\pi\epsilon_0\hbar v_{\rm F}} \tag{8}$$

(α_g is the "graphene fine structure constant", ~ 2).

Let's now analyze the structure of the wave functions and energy levels near a Dirac point in a magnetic field. Since the field necessary to generate a flux of the order

of ϕ_0 through a single plaquette of the honeycomb lattice is enormous ($\sim 10\,\mathrm{kT}$) at "reasonable" laboratory fields we can proceed as in the standard (GaAs-like) case and treat the momentum $\mathbf{p}$ which occurs in the energy spectrum (which is actually a crystal quasimomentum) as equivalent to a real momentum. The Hamiltonian matrix (eqn. (15) of lecture 21) is then modified by the standard "minimal-coupling" replacement $\mathbf{p} \rightarrow (\mathbf{p} - e\mathbf{A}(\mathbf{r})) \equiv \boldsymbol{\pi}$, i.e.

$$\hat{H} = \begin{pmatrix} 0 & v_{\mathrm{F}}\hat{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{\pi}} \\ v_{\mathrm{F}}\hat{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{\pi}}^{\dagger} & 0 \end{pmatrix} \tag{9}$$

We note that just as in the standard case the operators $\hat{\pi}_x$, $\hat{\pi}_y$ fail to commute:

$$[\hat{\pi}_x, \hat{\pi}_y] = ie\hbar B_z \tag{10}$$

At this point it actually becomes advantageous to re-define our "cyclotron frequency" (energy) so as to make it a function only of B and independent of n_s: thus, we define

$$\omega_0 \equiv v_{\mathrm{F}}/l_{\mathrm{M}} \tag{11}$$

It is easy to see, using $n_s = (2\pi l_{\mathrm{M}}^2)^{-1}$ that ω_0 is related to the ω_c defined above by

$$\omega_0/\omega_c = \frac{1}{2}\nu^{-1/2} \tag{12}$$

so that

$$V_c/\hbar\omega_0 = \alpha_g \tag{13}$$

independently of ν. From now on we can forget about ω_0, which has no real physical significance; as we may anticipate from (9) and (11), it is ω_0 which controls the energy spectrum of the Landau levels.

Just as in the standard case, we may solve for the energy eigenvalues and eigenfunctions in any convenient gauge (and, just as there, we expect that the explicit form of the eigenfunctions will depend on the gauge but the eigenvalues will not). It is convenient to consider a disk geometry.

It is straightforward to obtain the eigenvalues of the Hamiltonian (9). Let's rewrite it in terms of the cyclotron frequency ω_0 defined by eqn. (11): defining dimensionless operators $\hat{\Pi}_i \equiv l_{\mathrm{M}}\hat{\pi}_i/\hbar$, we have

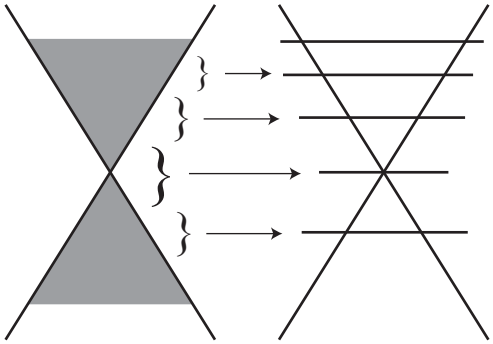
$$\hat{H} = \frac{\hbar\omega_0}{\sqrt{2}} \begin{pmatrix} 0 & \hat{\Pi}_x + i\hat{\Pi}_y \\ \Pi_x - i\hat{\Pi}_y & 0 \end{pmatrix} \tag{14}$$

with the commutation relations

$$[\hat{\Pi}_x, \hat{\Pi}_y] = i \tag{15}$$

We take the square of (14); using (15), the result is

$$\hat{H} = (\hbar\omega_0)^2 \begin{pmatrix} \hat{\Pi}_x^2 + \hat{\Pi}_y^2 + 1 & 0 \\ 0 & \hat{\Pi}_x^2 + \hat{\Pi}_y^2 - 1 \end{pmatrix} \tag{16}$$



But the expression $\hat{\Pi}_x^2 + \hat{\Pi}_y^2$, with the commutation relation (15), is just twice the Hamiltonian of the simple oscillator in units of $\hbar\omega_0$, ($\hat{\Pi}_x \to \hat{x}$, $\hat{\Pi}_y \to \hat{p}$), so we know that the possible eigenvalues are $2n+1$, $n = 0, 1, 2 \ldots$. Thus, we find the fundamental result for the eigenvalues E_n of $\hat{H}$

$$E_n = \pm\sqrt{n}\hbar\omega_0, \quad n = 0, 1, 2 \ldots \tag{17}$$

A crucial point to note is that while for $n \neq 0$ the eigenvalues come in pairs, one in the π^* band (+ sign) and one in the π band (− sign), *the eigenvalue with $n = 0$ is nondegenerate*. As we shall see below, each eigenvalue corresponds to a complete Landau level, and this includes the $n = 0$ case. Consequently, since on a scale large compared to $\hbar\omega_0$ the number of states must be conserved, we find that each Landau level "condenses" the original states in an energy region around it containing exactly N_ϕ states ($N_\phi \equiv$ number of flux quanta $= A/2\pi l_{\mathrm{M}}^2$). Because the density of states in zero field is proportional to ϵ, this means that the actual width in energy of the portion of the band so condensed varies roughly as $n^{-1/2}$. In particular, the $n = 0$ state condenses all the states *in both the π^* and the π band*, which have $|\epsilon| < \frac{1}{2}\hbar\omega_0$. The $n = 0$ state ("zero mode") is thus completely symmetric between the "electron" (π^*) and "hole" (π) states.

What do the actual wave functions of the different Landau levels look like? This is of course depends on the gauge. Let's consider a disk geometry and choose the radial gauge, in which the vector potential $\mathbf{A}(\mathbf{r})$ has the form $Br\hat{\boldsymbol{\phi}}$. Then, using the result $\partial_x \pm i\partial_y = e^{\pm i\phi}(\partial_r \pm i\partial_\phi)$, we find

$$\hat{H} = \hbar v_{\mathrm{F}} \begin{pmatrix} 0 & ie^{i\phi}(\partial_r + i\partial_\phi - r/l_{\mathrm{M}}^2) \\ ie^{-i\phi}(\partial_r - i\partial_\phi - r/l_{\mathrm{M}}^2) & 0 \end{pmatrix} \tag{18}$$

Evidently, the solution should have the form

$$\begin{pmatrix} f(r)e^{il\phi} \\ g(r)e^{i(l-1)\phi} \end{pmatrix} \tag{19}$$

At the present stage the detailed form of the functions $f(r)$ and $g(r)$ is not important: what is important is that, just as in the standard case, there is for each LL exactly one

solution associated with each value of l, which encloses (approximately) l quanta of flux. Hence the total number of states in each LL is exactly the number of flux quanta, just as in the standard case. Moreover, we can consider a Corbino-disk geometry and replay the Laughlin-Halperin argument: increase of the AB flux through the hole by one flux quantum pushes exactly one state across the disk, returning the system to its original state.

What are the consequences for the QHE? As in the standard case, we expect the latter to occur when the chemical potential is pinned between Landau levels by localized impurity states. This should happen close to the values of ν for which we get filling of an integral number of LL's. At this point we need to remember two things: First, since the $n = 0$ LL condenses only half of its levels from (say) π^* band, the points corresponding to exact filling of an integral number of LL's are shifted by $N_\phi/2$ from their values for the standard case. Secondly, what about the spin and valley degeneracies? The valley degeneracy is completely unsplit;[2] as to the spin degeneracy, everything seems consistent with the hypothesis that this is effectively unsplit in the fields ($\lesssim 15\,\mathrm{T}$) currently available. If that is correct, then the total degeneracy is 4, so one should find that the QH plateaux should occur close to the values of ν given by

$$\nu = 4n + 2 \tag{20}$$

where n can be positive, zero or negative; note that the plateaux at $\nu = 2$ and -2 correspond to the filling/emptying of the same LL, $n = 0$.

What about the value of the Hall conductance Σ_{xy} on the plateau? Intuitively, since we wish to get agreement with the classical result when ν is exactly $4n + 2 \equiv \nu_0$, this should be given by the same expression as in the standard case, namely

$$\Sigma_{xy} = \nu_0 e^2/h = (4n + 2)e^2/h \tag{21}$$

on the n-th plateau. The part involving $4n$ is straightforward, since all the LL's other than the lowest evidently transfer exactly 4 electrons per flux quantum. But what about the contribution of the "zero mode"? To discuss this we need to get into a little more detail on the form of the spinor components.

Using the correspondence of $\hat{\Pi}_x$, $\hat{\Pi}_y$ to the dimensionless coordinate $\hat{x}$ and momentum $\hat{p}$ of the simple harmonic oscillator, and the fact that the creation (annihilation) operator of the excitations (quanta) of the latter have the form $2^{-1/2}(\hat{x} - i\hat{p})$ $(2^{-1/2}(\hat{x} + i\hat{p}))$, we can rewrite the Hamiltonian (14) in the suggestive form

$$\hat{H} = \hbar\omega_0 \begin{pmatrix} 0 & a \\ a^\dagger & 0 \end{pmatrix} \tag{22}$$

(where the index which specifies the massive degeneracy within the LL is omitted, since it depends on the gauge). The form of the spinor energy eigenfunctions χ_n is in general

$$\chi_n = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_{n-1} \\ \psi_n \end{pmatrix}, \quad E_n = \sqrt{n}\hbar\omega_0 \tag{23}$$

[2] In the approximation of $A \rightleftharpoons B$ symmetry.

Of course, the actual expression for the spinor components $\psi_n(\mathbf{r})$ depends on the gauge; e.g. for the radial gauge they have the form $\psi_n(\xi) = H_n(\xi)\exp -\xi^2/2$ where H_n is a Hermite polynomial and ξ is, apart form a factor, $r - r_l$ where r_l is the radius which encloses l flux quanta. Recalling that the two spinor components correspond to the amplitudes to be on the A and B sublattices respectively, we see that for $n \neq 0$ we have equal amplitudes to be on each sublattice.

But what about the case $n = 0$? The function ψ_{-1} does not exist, so in this case we have simply

$$\chi_0 = \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix} \tag{24}$$

i.e. the state is *entirely localized on the B sublattice!* At first sight this looks strange, since at a fundamental level the A and B sublattices should be completely equivalent. However, they are not equivalent once we specify that the wave vector of interest is close to (say) Dirac point K rather than K' and *also* apply a magnetic field; this is clear from the fact that the helicity of the original Hamiltonian matrix element $\boldsymbol{\sigma} \cdot \mathbf{p}$ may be either parallel or antiparallel to the field. When we go from the Dirac point K to point K', the effect is to reverse the "intrinsic" helicity, and hence to interchange a and $a^\dagger$ in eqn. (22); the result is that the zero mode associated with the valley K' is entirely localized on the A sublattice, thereby restoring the symmetry.

The crucial question now is: If the $n = 0$ LL is filled, how much does it contribute to the Hall conductance? In the original experimental paper it is argued that since in all the $n \neq 0$ LL's both pseudospin states are populated whereas in the $n = 0$ level only one is, the density of states in the latter is 1/2 that of the other LL's and thus it contributes only $1/2(e^2/h)$ per spin and valley. This argument does not seem entirely satisfactory since any given LL function is normalized to 1 independently of whether it involves one or two pseudospin components. A second argument, which is given in various theoretical papers, is that for any given value of n other than zero there are *two* solutions ($\psi_{n-1} = \pm\psi_n$) while for $n = 0$ there is only one, and "therefore" the $n = 0$ level contributes only half the usual e^2/h per spin/valley. The problem with this argument is that for $n \neq 0$ one of the solutions has negative energy, and thus is already completely filled at zero doping; however, we do not want a nonzero conductance at this point! A third argument[3] rests on explicit consideration of the edge states; unfortunately, while for one type of edge (the so-called "zigzag" type, see next lecture) this argument produces quantization at $(n + 1/2)(e^2/h)$ per spin/valley, for the other ("armchair") type it gives the conventional result ne^2/h. It is rather puzzling that the result should be so sensitive to the exact nature of the edges, particularly since for a sample of "ragged" shape this is not necessarily uniformly defined.

My own tentative interpretation of the situation is as follows. Imagine first, that in the absence of a magnetic field the spectrum near the Dirac point has a small gap $\Delta\epsilon$ (such as might, for example, be produced by a week spin-orbit interaction, cf. lecture 21), and we consider the situation at zero doping and $T = 0$. By the standard textbook arguments, the system is an ordinary band insulator, i.e. $\Sigma_{xx} = \Sigma_{xy} = 0$. Next, suppose

[3] Peres et al., Phys. Rev. **B73**, 195411 (2006).

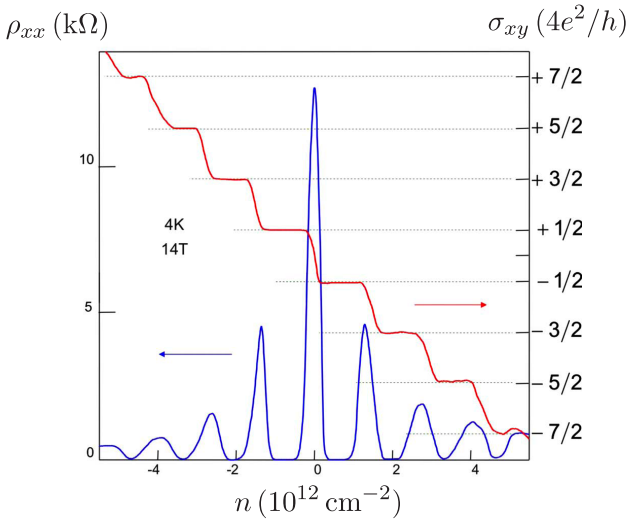


Figure 1: (Adapted from Novoselov et al., 2005.)

that we switch on a *weak* magnetic field, such that $\hbar\omega_0 \ll \Delta\epsilon$. This should produce no mixing-in of the conduction band states, so the many-body wave function remains an antisymmetrized Slater determinant of the valence-band states in which every state is occupied; since we can equally well write this as a Slater determinant of localized (Wannier) functions, both Σ_{xx} and Σ_{xy} must remain zero. Now comes the tricky step: we imagine turning $\Delta\epsilon$ down to zero. We know that now the energy eigenstates certainly form Landau levels, and that each LL will be formed of a superposition of the original $(B = 0)$ eigenstates corresponding to the appropriate range around $E_n \equiv \pm\sqrt{n}\hbar\omega_0$. Thus, the LL's with $n < 0$ will all be linear superpositions of the valence-band states. On the other hand, the level with $n = 0$ is formed half from valence-band and half form conduction-band states. The valence-band half, when combined with the contributions of the $n < 0$ levels, forms just another representation of the filled valence band, which by our earlier argument contributes nothing to Σ_{xy} (or Σ_{xx}); thus, when we start doping, it is only the "conduction-band" half of the zero mode which contributes to Σ_{xy}, giving the by now familiar $1/2\,(e^2/h)$ per spin/valley.[4]

It would be interesting to study how the behavior of Σ_{xy} evolves from the regime $\Delta\epsilon \gg \hbar\omega_0$ (where we get the standard "semiconductor" QHE result ne^2/h) to the regime $\Delta\epsilon \ll \hbar\omega_0$ where the above argument suggests we get the anomalous $(n + 1/2)e^2/h$ behavior; I suspect that the apparent "crossover" at $\Delta\epsilon = \hbar\omega_0$ (see Problem) may be related to the calculated difference between the behavior of samples with "zigzag" and "armchair" edges.

While there may be some open questions concerning the theory of the QHE in single-layer graphene, on the experimental side the results are beautifully clear-cut. In their original paper, Geim and co-workers measured the longitudinal resistivity and Hall con-

[4]This argument may actually be easier to grasp if we consider *hole* doping, i.e. empty the $n = 0$ level.

ductivity in a perpendicular field of 14 T at 4 K, varying the gate voltage so as to vary the carrier concentration between $\pm 5 \times 10^{12}\,\mathrm{cm}^{-2}$. The longitudinal sheet resistance (=resistivity) is symmetric around $n = 0$, with a zero value around the filling factors $\nu = 4n + 2$ and a sharp *peak* around $\nu = 0$, with a value of about $12\,\mathrm{k}\Omega$ ($\sim 2h/e^2$) (we will return to this in the next lecture). For the Hall conductance they found an antisymmetric pattern with plateaux around the filling factors $\nu = 4n+2$; around $\nu = 0$ Σ_{xy} is linear in ν, with a zero value at $\nu = 0$. Thus the data appear to be in very good agreement with (the bulk of) the theoretical expectations; the only at first sight surprising feature is that the plateaux are so wide, since the samples used were apparently rather pure ($\mu \approx 15000\,\mathrm{cm}^2/\mathrm{V\,sec}$). It should be noted that no special care was apparently taken with the edges of the sample, so that it is not clear whether these were "zigzag", "armchair" or mixed.

As the temperature is raised, the very precise QHE behavior seen at low temperatures gets blurred; e.g. in the longitudinal resistance the zeroes are filled in and the peaks get more rounded, leading to the (2D) Shubnikov-de Haas effect. In a typical 3D metal, this and related (e.g. dHvA) effects require temperatures $\sim$ a few K; however, in graphene the lowest SdH feature is still visible at room temperature.

We now turn to bilayer graphene. As we have seen, in the simplest model (with only $\tilde{A} \leftrightharpoons B$ hopping) the Hamiltonian for states close to a Dirac point has the form

$$\hat{H} = \frac{v_\mathrm{F}^2}{\gamma_1} \begin{pmatrix} 0 & (\hat{p}_x + i\hat{p}_y)^2 \\ (\hat{p}_x - i\hat{p}_y)^2 & 0 \end{pmatrix} \tag{25}$$

What QHE behavior, if any, does one expect for such a Hamiltonian? Making the standard replacement $\hat{\mathbf{p}} \to \hat{\mathbf{p}} - e\mathbf{A} \equiv \hat{\boldsymbol{\pi}}$ and introducing the dimensionless quantity $\hat{\mathbf{\Pi}}$ as previously, we find (where ω_0 now $\equiv v_\mathrm{F}/\gamma_1^{1/2}$)

$$\hat{H}^2 = \frac{(\hbar\omega_0)^2}{4} \begin{pmatrix} (\hat{\Pi}_x^2+\hat{\Pi}_y^2+1)^2+2(\hat{\Pi}_x+\hat{\Pi}_y+1) & 0 \\ 0 & (\hat{\Pi}_x^2+\hat{\Pi}_y^2-1)^2-2(\hat{\Pi}_x+\hat{\Pi}_y-1) \end{pmatrix} \tag{26}$$

$$= (\hbar\omega_0)^2 \begin{pmatrix} (n+1)(n+2) & 0 \\ 0 & n(n-1) \end{pmatrix} \tag{27}$$

($n = 0, 1, 2 \ldots$) so that the Landau-level energies are given by (the upper eigenvalue in eqn. (27) simply re-parametrizes the lower one)

$$E_n = \pm\hbar\omega_0\sqrt{n(n-1)} \tag{28}$$

The levels corresponding to $n \geqslant 2$ are nondegenerate, but the eigenvalue $E = 0$ is doubly degenerate, corresponding to $n = 0$ and 1. Consequently, it condenses states from *twice* the range addressed by the $n \geqslant 2$ states. Filling of a LL now occurs, for one spin and valley, at $\nu = \pm n$ (*not* including 0), and correspondingly after taking account of the spin and valley degeneracy the Hall conductance takes quantized values $4ne^2/h$ around values of $\nu = n$ =integer$\neq 0$. (By the same argument as for the single-layer case, the contribution of the doubly-degenerate zero mode is only its positive half, so

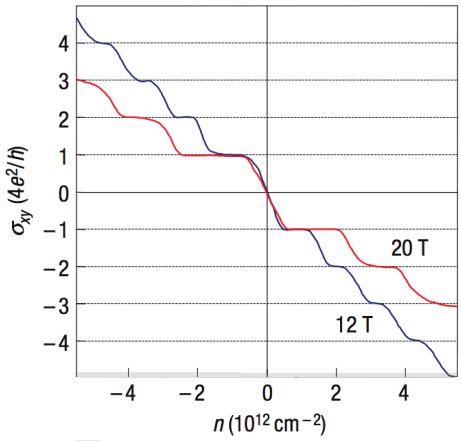


Figure 2: (Adapted from Novoselov et al., 2006.)

we correctly find that at $\nu = 1$ exactly we recover $\Sigma_{xy} = e^2/h$ in agreement with the classical expectation. This is exactly what is seen experimentally (see Novoselov et al., Nature Physics **2**, 177 (2006)).

The above simple picture refers to experiments using fields below 20 T. However, by now experiments have been conducted on single-layer graphene up to 45 T, and some interesting results are found:[5] Above 20 T, while the usual prominent plateaux are found at $\nu = 4(n + 1/2)e^2/h$, extra (weaker) plateaux are found at $\nu = 0, \pm 1$ and ± 4 (the one at $\nu = +4$ is barely visible in the data), in each case with $\Sigma_{xy} = \nu(e^2/h)$. It is possible to interpret these results within the framework of the above considerations about the Landau levels if we allow for the possibility of spin and valley splitting, as follows: Suppose the $n = 1$ level is spin-split by the Zeeman term (but not valley-split). Then, starting from $\nu = 2$ (the zero LL completely filled) we will first fill the spin-up $n = 1$ state, then there will be a gap before we start filling the spin-down $n = 1$ state (just as in the standard IQHE at $\nu = 1$). Since the spin-up $n = 1$ state contains 2 species (from the two valleys), its complete filling corresponds to $\nu = 4$.

So far, so good, but what about the plateaux at $\nu = 0$ and ± 1? (The $\nu = 0$ plateau is particularly interesting, since in 45 T it can be seen, with $R_{xy} \to 0$ but R_{xx} *peaking* at $\sim 30\,\mathrm{k}\Omega$, even at room temperature, see the cited reference). The $\nu = \pm 1$ plateaux can be explained as above if the spin degeneracy is lifted (one would then expect $\nu = (2n + 1)(e^2/h)$). However, the $\nu = 0$ plateau is more problematical; it would apparently require in a noninteracting-electron picture, not just that the valley degeneracy is also lifted (by itself this would lead to $\nu = (n + 1/2)(e^2/h)$) but rather that the sublattice degeneracy be lifted *by an amount comparable to* $\hbar\omega_0$, (so that we recover "semiconductor" picture, cf. above). This seems very difficult to understand within a single-particle picture; consequently, the general belief is that the occurrence of this state is evidence for interaction effects. We will return to this question in the next

[5] Zhang et al., PRL **96**, 136806 (2006).

lecture.

With this single exception, the QH effects we have discussed so far, while "anomalous", are analogous to the integral QHE in standard system, in the sense that they have a natural explanation in terms of noninteracting electrons. What about the fractional QHE? This was expected theoretically to occur in single-layer graphene, but for some years proved elusive. Very recently (Oct. 14, 2009!) the group of E. Andrei at Rutgers has reported evidence[6] for a FQHE at $\nu = -1/3$ (not $-4/3$!). They study suspended sheet of single-layer graphene with a mobility (at $n_s \approx 10^{10}\,\mathrm{cm}^{-2}$)[7] of $2.6\times10^5\,\mathrm{cm}^2/\mathrm{V\,sec}$, with areal density in the range 10^9-$10^{10}\,\mathrm{cm}^{-2}$ and fields up to 12 T; temperature range from ∼1-80 K. The measurements, unlike most of those on the QHE, are two-terminal, which means that it is impossible in general to separate the effects of R_{xx} and R_{xy}; however, theory suggests that the plateaux which in the standard 4-terminal setup would appear in R_{xy} should appear, in this geometry, in R_{xx}, and this expectation is confirmed in that they see plateaux corresponding to the standard values $\nu = 4(n + 1/2)(e^2/h)$, as well as (above 2 T) $\nu = -1$ and $\nu = 3$, which are presumably due as above to the lifting of spin degeneracy.

Apart from some very intriguing observations in the neighborhood of $\nu = 0$, which will be discussed in the next lecture, the major result reported in this paper is the observation of a plateau in the measured resistance at $\nu = -1/3$; although it is less pronounced than those at $\nu = 1$ and, especially at the "standard" value $\nu = 2$, it seems definite. The interpretation as a genuine QHE plateau is strengthened by the observation that as the temperature is raised it "dies" around 20 K, which is within 8% of the predicted gap against excitation of $e^*/3$ quasiparticle in a Laughlin-type $\nu = 1/3$ state.

An even more recent experiment[8], by the group of Philip Kim at Columbia, also reported evidence for a FQHE in single-layer graphene. Like the Rutgers group, they studied a suspended sample with two-terminal contact in fields up to 14 T, but worked at somewhat higher areal densities ($\lesssim 3\times10^{11}\,\mathrm{cm}^{-2}$). They find the standard $\nu = 4(n+1/2)$ plateaux (also $\nu = 1$), and also a distinct plateau at $\nu = 0.30 \pm 0.02$, which survives up to $T \sim 10$ K. In addition, they find features at $\nu = 0.46 \pm 0.02$ and $\nu = 0.68 \pm 0.05$; they tentatively identify the latter as a $\nu = 2/3$ QH state but are less confident about the $\nu = 1/2$ feature. Finally, like the Rutgers group, they obtain some interesting data around $\nu = 0$ which I will mention in the next lecture.

[6] X. Du et al. Nature online (Oct. 14 2009).

[7] Recall that in graphene the mobility is a strong function of n_s ($\propto n_s^{-1/2}$).

[8] K. Bolotin et al., Nature online (1 Nov. 2009).