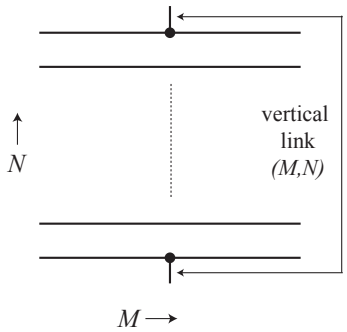
The Kitaev models

Since many of the ideas involved in TQC appear rather abstract and perhaps a priori speculative, it is useful to have a concrete model that is analytically soluble, so that one can see quite explicitly how these features converge. In the (short) history of the subject two models, both due to A.Yu. Kitaev, have been particularly influential. While to date neither has been fully implemented, there is a reasonable hope of implementing them using purpose-engineered systems such as optical lattices or arrays of Josephson junctions. The first model, the so-called "toric code" is a system that has degenerate topologically protected groundstates, and this is in principle suitable for the implementation of a quantum memory; however, as in the case of the $\nu = 1/3$ QHE on a torus, the excitations are *abelian* anyons, so it is not directly useful for TQC. Its advantage is that the analytic solution is rather straightforward. The second model, the so-called "Kitaev honeycomb model" sustains nonabelian (Ising) anyons over part of the parameter space, so is in principle suitable for (partially protected) TQC, but is considerably more difficult to analyze. I will discuss them in turn, but in the case of the honeycomb model will have to quote some of the results without derivation.

The toric code

Consider a 2D plane lattice of $N \times N$ points, with periodic boundary conditions (equivalent to bending the plane into a torus). We define a spin vector $\boldsymbol{\sigma}_i$ ("qubit") associated with each *link* (not site!) of the lattice; thus the total number of spins (qubits) is $2N^2$.



Note that because of the periodic boundary conditions, these links include (e.g.) one between the two sites $(M, 1)$ and (M, N) which lie on opposite faces of the array. If we label each vertical link by convention by the site at its bottom end, then this link is labeled (M, N); it has exactly the same status as any other vertical link. Similar remarks apply to the horizontal links.

All components of spins of different links are mutually commuting, while the different Cartesian components of spin on a given link anticommute, i.e.

$$\{\hat{\sigma}_{i\alpha}, \hat{\sigma}_{i\beta}\} = 2\delta_{\alpha\beta} \qquad (\alpha, \beta = x, y, z) \tag{1}$$

but

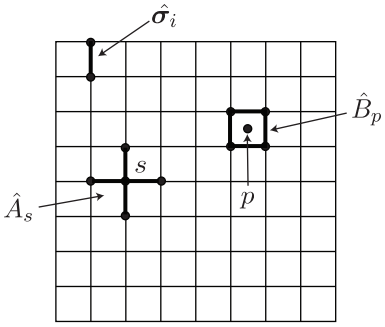
$$[\hat{\sigma}_{i\alpha}, \hat{\sigma}_{i\beta}] = 0 \text{ if } i \neq j \tag{2}$$

As a result of (1) and (2) we have the important relation

$$[\hat{\sigma}_{i\alpha}\hat{\sigma}_{j\alpha},\hat{\sigma}_{i\beta}\hat{\sigma}_{j\beta}] = 0 \qquad \forall\, i,j \tag{3}$$

Consider the (not very physical-looking) Hamiltonian

$$\hat{H} = -\sum_s \hat{A}_s - \sum_p \hat{B}_p \tag{4}$$



where

$$\hat{A}_s \equiv \prod_{j\epsilon \text{ star}(s)} \hat{\sigma}_j^x, \qquad \hat{B}_p \equiv \prod_{j\epsilon \text{ bound } (p)} \hat{\sigma}_j^z \tag{5}$$

Here the "star" of the lattice point S means the four links emanating from S, and the "boundary" of the *intercalated* "pseudo-lattice point" p means the four links surrounding p. Note that the possible eigenvalues of both $\hat{A}_s$ and $\hat{B}_p$ are ± 1, and that

$$\prod_s \hat{A}_s = \prod_p \hat{B}_p = +1 \tag{6}$$

The condition (6) is very important in the subsequent analysis: note that it follows only because we have imposed periodic boundary conditions, so that (for example) $\boldsymbol{\sigma}_{M,N+1}$ is identified with $\tilde{\sigma}_{M,1}$, thus every link is represented exactly twice in each of the two products.

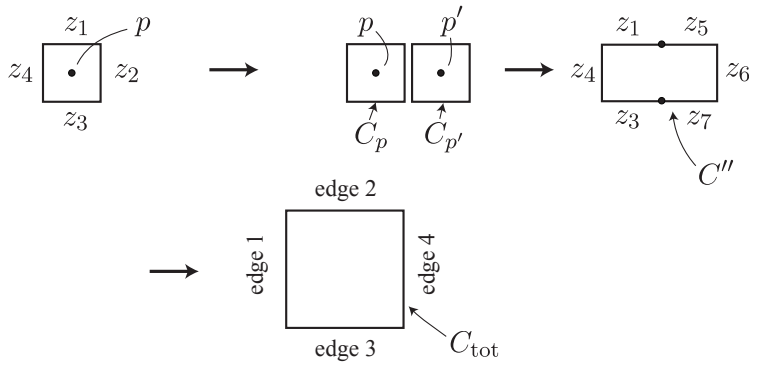
It is obvious that $[\hat{A}_s, \hat{A}_{s'}] = [\hat{B}_p, \hat{B}_{p'}] = 0,\ \forall s, s', p, p'$. Moreover, since A_s and B_p have either 0 or 2 links in common, it follows from (3) that $[\hat{A}_s, \hat{B}_p] = 0,\ \forall s, p$. Hence all the $\hat{A}_s$ and all the $\hat{B}_p$ are good quantum numbers (that is, they can be diagonalized simultaneously with $\hat{H}$ and with one another), so that their values can be used to label the energy eigenstates. In particular, the groundstate is a common eigenstate of all the $\hat{A}_s$ and all the $\hat{B}_p$, with $\hat{A}_s = \hat{B}_p = +1,\ \forall s, p$; it has energy $E_0 = -4N^2$. Furthermore, in view of the relation (6), any excited state must have at least two A_s or two B_p reversed, thus the minimum excitation energy is given by

$$E_{\min} = 4 \tag{7}$$

Now comes the crunch: The groundstate must be 4-fold degenerate! The easiest way to see this is to note that since there are $2N^2$ qubits, the total Hilbert space must have dimension 2^{2N^2}. However, in view of the constraint (6), there are only $2N^2 - 2$ independent A_s and B_p, thus only $2N^2 - 2$ conditions on the GS. Hence the dimensionality of the GS manifold is $2N^2 - (2N^2 - 2) = 2^2 = 4$.

This argument, however, does not give us much intuition into the nature of the degeneracy. To get some, let's work in the standard ("σ_z") basis and associate with $\sigma_j^z = +1(-1)$ the quantity $z_j = 0(1)$. Then the GS constraint $B_p = 1$ is equivalent to

$$\varphi_p \equiv \sum_{j\epsilon C_p} z_j = 0 \qquad (\text{mod } 2) \tag{8}$$



where C_p is the boundary of pseudo-lattice point p. Now imagine combining two neighboring blocks ("plaquettes"); and consider

$$\varphi_{C''} \equiv \sum_{j \in C''} z_j \quad (\equiv z_1 + z_4 + z_3 + z_7 + z_6 + z_5) \tag{9}$$

Evidently, since have removed *twice* z_2, we have in the GS

$$\varphi_{C''} \equiv \varphi_p + \varphi_{p'} - 2z_2 = 0 \pmod 2. \tag{10}$$

Iterating this procedure, we have for the φ corresponding to the whole array,

$$\varphi_{\rm tot} \equiv \sum_{j \epsilon C_{\rm tot}} z_j = 0 \pmod 2 \tag{11}$$

But in view of the periodic boundary conditions[1] we have edge 1 ≡ edge 3, edge 2 ≡ edge 4. Hence the groundstate can be characterized by two independent quantities, each of which can take only the values 0, 1:

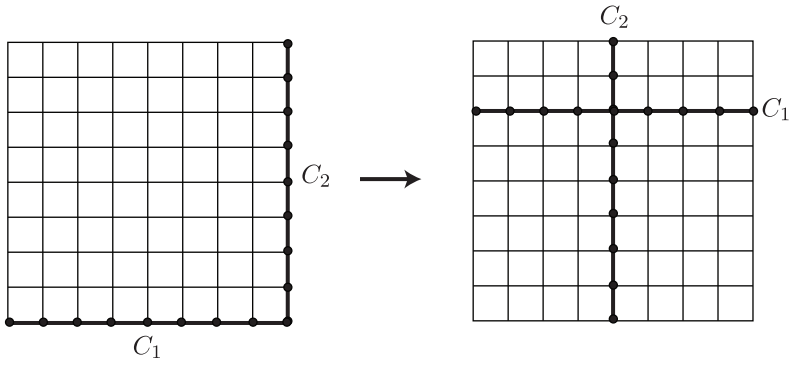
$$\upsilon_1 \equiv \sum_{j \epsilon \text{ edge1}} \hat{z}_j = 0, 1 \tag{12a}$$

$$\upsilon_2 \equiv \sum_{j \epsilon \text{ edge2}} \hat{z}_j = 0, 1 \tag{12b}$$

giving 4 different states. Of course, we could equally well have defined quantities

$$\hat{\xi}_1 \equiv \prod_{j \epsilon \text{ edge 1}} \hat{\sigma}_j^z = \pm 1, \text{ etc.} \tag{13}$$

but the outcome is the same. Furthermore, it is clear that in the definition of $\hat{\upsilon}_1, \hat{\upsilon}_2$ we can displace the contours to arbitrary values of n (for υ_1) and of m (for υ_2) provided that

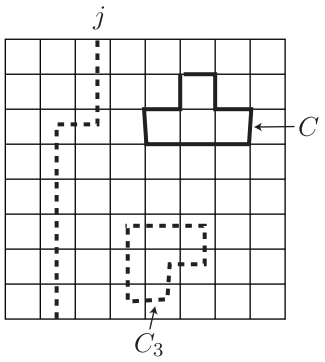


they still loop around the torus.

Now let us raise the question: What operators connect the degenerate states in the GS manifold? Let's continue to work in the standard (σ^z-) representation, and define "string" operators of the form

$$Z_C \equiv \prod_{j\epsilon C} \hat{\sigma}_j^z, \qquad X_{C'} \equiv \prod_{j\epsilon C'} \hat{\sigma}_j^x \tag{14}$$

where C and C' denote *closed* contours, i.e. they return to their starting point (but may or may not loop around the torus in the process). It is convenient to imagine C' to be displaced to the intercalant sites[2] or in the figure, which shows a "contractible" C loop (i.e. one that does not wind around the torus in either direction, and both contractible and non-contractible C' loops.



If a Z-loop is contractible, then according to the above argument it can be written in the form $\prod_{p \text{ inside } C} B_p$, and hence within the GS manifold is equivalent to the unit operator. Similarly, a little thought shows that any contractible X-loop can be written as $\prod_{S \text{ inside } C'} A_s$, and again is equivalent to the unit operator within the GS manifold.

What if a loop is *not* contractible, but encircles the torus in (say) the horizontal direction? In the case of a Z-loop, such a loop is exactly the operator $\hat{\xi}_1$, so it has the appropriate eigenvalue ± 1 depending on whether v_1 is 0 or 1. Similarly, a Z-loop that encircles the torus in the vertical direction is equivalent to $\hat{\xi}_2$ and has eigenvalue ± 1 depending on whether v_2 is 0 or 1. From now on it is convenient to work in terms of operators $\hat{Z}_1$ and $\hat{Z}_2$, corresponding to Z-loops encircling the torus in the horizontal and vertical directions respectively, with eigenvalues ± 1. Evidently, $[\hat{Z}_1, \hat{Z}_2] = 0$, and the four possible combinations of eigenvalues $(\pm 1, \pm 1)$ define a "2-qubit" manifold. Now the crunch: what if an X-loop encircles the

[1]Strictly speaking, we need to interpret "edge 1" as corresponding to the horizontal link $(M, N+1)$ rather than (M, N), etc.

[2]This is just a notational convention that allows us to define the notion of "inside C'" unambiguously, see below.

torus in (say) a horizontal direction (call this operator $\hat{X}_1$)? It does not affect $\hat{Z}_1$, because it is always possible to displace the X- and Z-loops so that they have no link in common[3]

$$[\hat{X}_1, \hat{Z}_1] = 0 \tag{15}$$

and of course a similar relation for $\hat{X}_2$ and $\hat{Z}_2$. The situation is different as regards $\hat{Z}_2$; with even the simplest choice of paths, it is clear that the two loops must have one spin in common (and more complicated choices always given an odd number). Suppose we label this spin 0, then the *anti*commutator of $\hat{X}_1$ and $\hat{Z}_2$ is given by

$$\{\hat{X}_1, \hat{Z}_2\} = \{\hat{\sigma}_0^x, \hat{\sigma}_0^z\} = 0 \tag{16}$$

and similarly $\{\hat{X}_2,\ \hat{Z}_1\} = 0$. Thus, we can make the correspondences

$$\hat{Z}_1 \rightarrow \hat{\sigma}_1^z \tag{17a}$$

$$\hat{Z}_2 \rightarrow \hat{\sigma}_2^z \tag{17b}$$

$$\hat{X}_1 \rightarrow \hat{\sigma}_2^x \qquad \text{(note } 1 \rightleftarrows 2 \text{ here !)} \tag{17c}$$

$$\hat{X}_2 \rightarrow \hat{\sigma}_1^x \tag{17d}$$

so that the Z's and X's form Pauli matrices in the 2-qubit (4-dimensional) groundstate manifold.

So far, so good; we have shown that the groundstate of the toric-code Hamiltonian (4) is 4-fold degenerate, but we have not shown that the degeneracy is "topologically protected." By this we mean the following: Consider any operator $\hat{\Omega}$ that is of the form

$$\hat{\Omega} \equiv \hat{\sigma}_i^\alpha\ \hat{\sigma}_j^\beta\ \hat{\sigma}_k^\gamma \ldots \tag{18}$$

where the links i, j, k ..., while not necessarily "nearest-neighbors," are "nearby," in the sense that the maximum distance involved can be made arbitrarily small compared to N in the thermodynamic limit $N \rightarrow \infty$. Such operators may be called "local". Since we can displace the contours C_1, C_2, C_1', C_2' that define the operators Z_1, Z_2, X_1, X_2 arbitrarily (so long as they continue to wrap around the torus) it is clear that we can choose them so that $\hat{\Omega}$ and (e.g.) $\hat{Z}_1$ have no spins in common, so that

$$[\hat{\Omega}, \hat{Z}_1] = 0 \tag{19}$$

In a similar way we can show that $[\hat{\Omega}_1 \hat{Z}_2] = [\hat{\Omega}_1 \hat{X}_1] = [\hat{\Omega}_1 \hat{X}_2] = 0$. But recalling that $\hat{Z}_1 \rightarrow \hat{\sigma}_{Z1}$, $\hat{X}_2 \rightarrow \hat{\sigma}_{x1}$, etc., this means that within each of the "single-qubit" 2D Hilbert spaces $\hat{\Omega}$ commutes with both the Pauli operators $\hat{\sigma}_z$ and $\hat{\sigma}_x$. But it is easily shown that in

[3] If they do have links in common (i.e. cross one another), it is easy to see that it must be an even number, so according to eqn. (3) the operators $\hat{X}_1$ and $\hat{Z}_1$ still commute.

a 2D Hilbert space any operator having this property must be just a multiple of the unit operator, so it follows that within the GS manifold

$$\hat{\Omega} = \text{ const. } \hat{1}. \tag{20}$$

Thus, any effect by which $\hat{\Omega}$ causes transitions between the different groundstates, or shifts their relative energy, must come from indirect processes, in which $\hat{\Omega}$ generates a virtual transition out of the GS manifold. However, assuming that the number of operators in $\hat{\Omega}$ is $\mathcal{O}(1)$ (not $\mathcal{O}(N)$), it must operate of order N times (since the operators $\hat{Z}_i$ and $\hat{X}_i$ change the state of $\mathcal{O}(N)$ links); each operation involves an energy denominator $\sim E_0$, so if the relevant matrix element of $\hat{\Omega}$ is ω, the final transition amplitude is $\mathcal{O}(\omega/E_0)^N \sim \exp -\alpha N$ where $\alpha \equiv \ln(E_0/\omega)$. Thus the degeneracy of the groundstate is indeed "topologically' protected in the thermodynamic limit, provided only that $\omega < E_0$.

An attractive feature of the toric code model is that one can actually write down the wave functions of the four degenerate groundstates in explicit form. Consider the expression ("Hadamard" state)

$$\Psi_{\text{Had}}(z_1 z_2 \dots z_N) \equiv \prod_{j=1}^{2N^2} \hat{H}_j \, |0\rangle \tag{21}$$

where $|0\rangle$ is the state specified by $z_j = 0, \ \forall j$ and $\hat{H}_j$ is the Hadamard gate on the j-th qubit, with the explicit form

$$\hat{H} = 2^{-1/2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{22}$$

The "Hadamard" state (21) is just a linear superposition of all the possible 2^{2N^2} states of the system with identical amplitude; it is an eigenfunction of each of the $\hat{\sigma}_x^j$ with eigenvalue $+1$, thus also an eigenstate of all the operators $\hat{A}_s$ with eigenvalue $+1$. However, it is not an eigenfunction of the $\hat{B}_p$, since different terms have different values of $\hat{\sigma}_z^j$ and thus of $\hat{B}_p$. We, therefore, project the Hadamard state on to the subspace corresponding to $\hat{B}_p = 1, \ \forall p$. Because of the relation $[\hat{A}_s, \hat{B}_p] = 0$, this does not spoil the result that that $A_s = +1, \ \forall s$. Thus the resulting state is a linear combination of the four groundstates specified by (11), with equal amplitudes. Finally we select a particular groundstate by imposing the extra pair of conditions which specify υ_1 and υ_2 (eqn. (12)). The explicit expression for a particular state $\Psi(\upsilon_1 \upsilon_2)$ is therefore, apart from normalization,

$$\Psi(\upsilon_1, \ \upsilon_2) = \sum_{\{z_j\}} |z_1 z_2 \dots z_{2N^2}\rangle \tag{23}$$

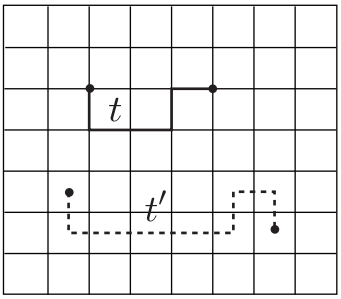
$$\forall B_p = +1, \ \sum_{j \epsilon \text{ edge } 1} z_j = \upsilon_1, \ \sum_{j \epsilon \text{ edge } 2} = \upsilon_2$$

(the condition $\forall B_p = +1$ is not noted explicitly in Kitaev's original paper, but is implicit in the preceding discussion.) It is superfluous to note that the states (22) are strongly entangled.

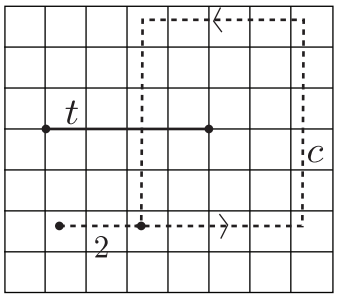
So much for the groundstate manifold. Now let's consider the elementary excitations of the model. Let's start with the system in the GS manifold and create "strings" $S^z(t)$, $S^x(t)$ defined by

$$\hat{S}^z(t) \equiv \prod_{j \epsilon t} \hat{\sigma}_j^z, \qquad \hat{S}^x(t') \equiv \prod_{j \epsilon t'} \hat{\sigma}_j^x \tag{24}$$

where now t and t' are *open* contours with ends (see figure). The operator $\hat{S}^z(t)$ obviously commutes with all the B_p, and it commutes with all the A_s except for the end-point sites, where it inverts the sign of A_s; consequently it costs an energy 4. Similarly, the operator $\hat{S}^x(t)$ commutes with all the A_s and with all the B_p except the end ones which it inverts; again the cost is 4. So a string of each type costs the minimum excitation energy 4. Actually, once the endpoints are fixed it doesn't matter how we lay the string, provided that we neither wrap it around the torus nor cross another string (see below). Thus it is only the endpoints that are significant, and *they can be regarded as independent quasiparticles.*[4]



It is clear that moving one X-qp (the end of a S^x-string) around a second X-qp (or a Z-qp around a Z-qp) has no effect (i.e. does not change the state), since all the operators involved commute. Also, moving a *pair* of X-qps around a pair of Z-qps has no effect (all spin operators occur either zero or 2 times). But what about moving a *single* X-qp around a *single* Z-qp? (see figure). This is equivalent to *crossing* a Z-string with an X-string, and will result in a single link j being operated on with both $\hat{\sigma}_j^x$ and $\hat{\sigma}_j^z$. If $S^z(t)$ and $S^x(t;')$ have no link in common, they commute, while if they have a single link in common (i.e. cross once) they anticommute. Hence, if $|\psi_q\rangle$ denotes the state created by $S^x(q)$ and $|0\rangle$ the GS



$$\begin{aligned} \Psi_{in} &= \hat{S}^z(t)\hat{S}^x(q)|0\rangle \equiv \hat{S}^z(t)\ |\psi_q\rangle \\ \Psi_f &= \hat{S}^x(c)\hat{S}^z(t)\hat{S}^x(q)|0\rangle \equiv \hat{S}^x(c)\hat{S}^z(t)\ |\psi_q\rangle \\ &= -\hat{S}^z(t)\hat{S}^x(c)\ |\psi_q\rangle \end{aligned} \tag{25}$$

But $\hat{S}^x(c)|\psi_q\rangle \equiv \hat{S}^x(c)\hat{S}^x(q)\ |0\rangle$, and since the closed-contour operator $\hat{S}^x(c)$ can be shrunk to the identity, this is just $S^x(q)\ |0\rangle \equiv |\psi_q\rangle$. Hence

$$\Psi_f = -\hat{S}^z(t)\ |\psi_q\rangle = -\Psi_{in} \tag{26}$$

Hence, where a Z-qp is encircled by an X-qp there is a sign change $-1 \equiv \exp i\pi$, and correspondingly when an X-qp and a Z-qp are interchanged we get a factor $\exp i\pi/2$ i.e. X and Z are "relative semions" (note *not* "fermions"!)

[4] Compare the idea of separating a (number-conserving) excitation of a degenerate Fermi liquid into two independent "particle" and "hole" components.

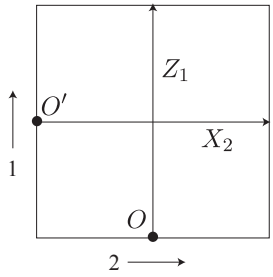
What is the connection between the topologically protected groundstate and the anyonic statistics of the excitations? In considering the operators X_1, X_2, Z_1, Z_2 necessary to move between the degenerate groundstates we showed that

$$\{\hat{Z}_1, \hat{X}_2\} = 0 \tag{27}$$

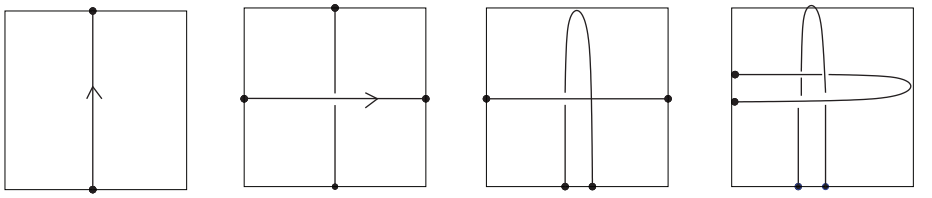
which implies

$$\hat{X}_2^{-1}\hat{Z}_1^{-1}\hat{X}_2\hat{Z}_1 = -1 \tag{28}$$

We can interpret the operation $\hat{Z}_1$ as creating a pair of anyons at O and taking *one* of them around the torus in the vertical direction back to O; the operation $\hat{Z}_1^{-1}$ then reverses the encirclement and annihilates the anyon pair back into the groundstate. Similarly $\hat{X}_2$ creates a pair at O' and takes *one* of the anyons around in the horizontal direction.

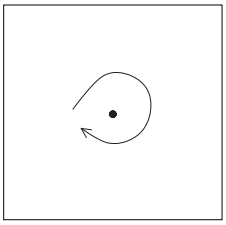


However, the operation (28) does not actually need a torus:



$\hat{Z}_1$ $\hat{X}_2\hat{Z}_1$ $\hat{Z}_1^{-1}\hat{X}_2\hat{Z}_1$ $\hat{X}_2^{-1}\hat{Z}_1^{-1}\hat{X}_2\hat{Z}_1$

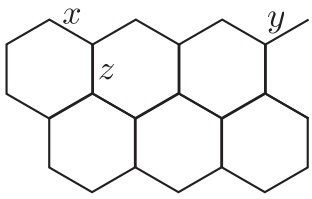
This is topologically equivalent to



so we would expect the "encirclement phase" to be π and the "exchange phase" $\pi/2$, as already found. (Note that this argument does not as it stands distinguish between $+\pi/2$ and $-\pi/2$!)

Because of the form of the Hamiltonian (eqn. (4)), which involves 4-spin interactions, it seems rather unlikely that the toric code model is implemented by any naturally occurring physical system; and while it is not out of the question that it may be possible to implement it in purpose-engineered systems such as optical lattices, it will almost certainly not be at all trivial to do so. However, it fortunately turns out that it is actually isomorphic to a particular parameter regime of a second model, the "Kitaev honeycomb model," which may be easier to implement experimentally. I now turn to this topic, though unlike in the toric case, I shall have to quote a number of results without derivation.

The Kitaev honeycomb model[5]



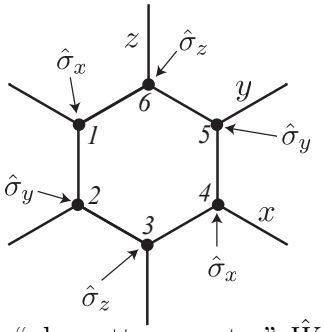
We consider a planar honeycomb lattice with quantum spins of 1/2 sitting on each *vertex* (not, as in the toric code, on the links). The links are labeled x, y, z according to their orientation, as shown. The Hamiltonian is of the form of a 2-spin nearest-neighbor interaction, with the components that are coupled depending on the direction of the nearest-neighbor link in question:

$$\hat{H} = -J_x \sum_{x-\text{links}} \hat{\sigma}_i^x \hat{\sigma}_j^x - J_y \sum_{y-\text{links}} \hat{\sigma}_i^y \hat{\sigma}_j^y - J_z \sum_{z-\text{links}} \hat{\sigma}_i^z \hat{\sigma}_j^z \tag{29}$$

where the coupling constants J_x, J_y, J_z may have either sign. We note (and will return to this and related points later in the context of possible implementations) that the association of (e.g.) the σ^x- components with an "x-link" is purely conventional.

In analyzing the behavior of the model described by eqn. (29) one makes essential use of a generalized form of eqn. (3), which we recall is

$$[\hat{\sigma}_{i\alpha}\hat{\sigma}_{j\alpha},\ \hat{\sigma}_{i\beta}, \hat{\sigma}_{j\beta}] = 0 \tag{30}$$
$$(i, j = \text{sites},\ \alpha, \beta = \text{ Cortesian components})$$



We can generalize this to

$$[\hat{\sigma}_{i\alpha}\hat{\sigma}_{j\beta},\ \hat{\sigma}_{i\gamma}\hat{\sigma}_{j\gamma}] = 0 \tag{31}$$

provided that α and β are both different from γ. Consider the "plaquette operator" $\hat{W}_p$ defined by

$$\hat{W}_p \equiv \hat{\sigma}_1^x\ \hat{\sigma}_2^y\ \hat{\sigma}_3^z\ \hat{\sigma}_4^x\ \hat{\sigma}_5^y\ \hat{\sigma}_6^z \tag{32}$$

[5] A. Yu Kitaev, Ann. Phys. **321**, 2 (2006); H-D Chen and Z. Nussinov, J. Phys. A: Math. Theor. **41**, 075001 (2008).

Because each spin component acts only at a vertex where the external bond is in the corresponding direction, $\hat{W}_p$ commutes with all the terms in (29) that involve bonds external to the plaquette. Also, because of (31), it commutes with the terms involving the bonds that form the boundary to the plaquettes. Moreover, the different W_p commute with one another, since any two plaquettes share either 0 or 2 vertices. Consequently, the W_p's can be independently diagonalized, with eigenvalues

$$W_p = \pm 1. \tag{33}$$

The total Hilbert space can thus be broken up into manifolds (sectors) defined by different values of the set $\{W_p\}$. If we rewrite W_p as

$$\hat{W}_p \equiv \hat{\sigma}_1^x \, \hat{\sigma}_4^x \, \hat{\sigma}_2^y \, \hat{\sigma}_5^y \, \hat{\sigma}_3^z \, \hat{\sigma}_6^z \tag{34}$$

it is plausible (and actually turns out to be correct) that for J_x, J_y, $J_z > 0$ ("ferromagnetic" interactions) the groundstate lies in the sector $W_p = +1, \ \forall p$. All states in sectors with any of the $W_p = -1$ have nonzero energy gaps relative to the GS.

However, the original Hilbert space is 2^n-dimensional, where n is the number of *vertices*, while the number of degrees of freedom described by the $\{W_p\}$ is the number of *plaquettes*, which is $n/2$ not n. Hence each sector of the Hilbert space, and in particular the sector containing the GS, is $2^{n/2}-$ dimensional (crudely speaking, each plaquette p possesses one DOF that is not described by W_p).

The complete solution of the honeycomb model is not at all trivial; the most accessible discussion (in my opinion) is given by Chen and Nussinov, ref. cit. They show that within the groundstate sector $W_p = +1, \ \forall p$, the Hamiltonian can be mapped formally to that of a 2D p-wave Fermi superfluid with Hamiltonian

$$\hat{H} = \sum_q \epsilon_q a_q^\dagger a_q + \frac{i\Delta_q}{2}(a_q a_{-q}^\dagger + \text{H.c.}) \tag{35}$$

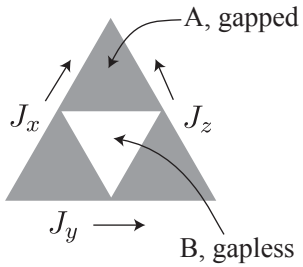
with

$$\begin{aligned} \epsilon_q &\equiv 2J_z - 2J_z \cos q_x - 2J_y \cos q_y \\ \Delta_q &\equiv 2J_x \sin q_x + 2J_y \sin q_y \end{aligned} \tag{36}$$

We find by the standard BCS procedure that the quasiparticle excitation spectrum has the usual form

$$E_q = \sqrt{\epsilon_q^2 + \Delta_q^2} \tag{37}$$

It is easy to verify that if the magnitude of any of the J_α ($\alpha = x, y, z$) is greater than the sum of the other two, then irrespective of the sign of the J's the quasiparticle spectrum is gapped. Since we already saw that there is a gap between the GS sector and any other, this implies that in this parameter regime there is a nonzero gap for *any* excitation of the groundstate ("gapped" phase). This regime is called the A phase of the model.



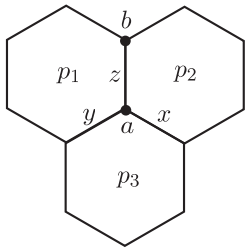
Suppose we start in the GS sector with no fermionic (i.e. BCS-like) excitations present, and try to create other types of excitation. The obvious possibility is to change the values of some of the W_p from $+1$ to -1, thereby creating a "vortex" on plaquette p. However, because of the obvious constraint $\prod_p W_p = +1$, we can create these "vortices" only in pairs. One possibility is to apply the operator (see figure)

$$\exp -i\frac{\pi}{2}\hat{\sigma}_a^z \tag{38}$$

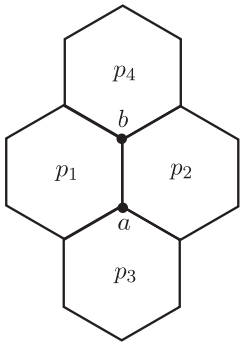
Since this leaves σ_a^z invariant but inverts σ_a^x and σ_a^y, it changes the value of W_p on both p_1 and p_2 but leaves all other plaquettes unaffected.

A second possibility is to apply (see figure) the operators

$$\exp i\pi\hat{\sigma}_a^x \exp i\pi\hat{\sigma}_b^y \tag{39}$$



This changes 2 spins on each of the plaquettes p_1 and p_2, so does not affect W_{p_1} or W_{p_2}, but inverts only one spin on p_1 and p_4, thus effecting $W_{p_1} \to -W_{p_1}$, etc. Once a vortex has been created, it can be moved around just like the anyons in the toric-code model. In fact, one can make an *exact mapping* between the zero-fermion sector of the honeycomb model in the "A" (gapped) phase and the toric code. An obvious consequence of this is that the anyons of the A phase (the vortices) are *abelian*, (something that is not too difficult to demonstrate explicitly) and thus cannot be used for quantum computation.



The more interesting phase in the context of TQC is the B ("gapless") phase, which occupies the parameter regime

$$\begin{aligned} |J|_x &\le |J_y| + |J_z| \\ |J|_y &\le |J_z| + |J_x| \\ |J|_z &\le |J_x| + |J_y| \end{aligned} \tag{40}$$

i.e. the unshaded region of the figure. This is in some sense the regime where one expects maximal frustration and hence perhaps an interestingly entangled groundstate. The general belief, based on somewhat indirect arguments, is that the excitations in this regime are *nonabelian anyons*; but to the best of my knowledge a direct and explicit demonstration of this is still lacking.[6] In any case, this phase as it stands is useless for TQC, because in view of the gaplessness of the excitation spectrum the usual arguments about protection from indirect processes fail. However, it turns out that application of a magnetic field $\boldsymbol{h}$ results in a gap of the form

$$\Delta = \text{const. } h_x h_y h_z / J^2 \tag{41}$$

so that in principle TQC should be feasible.

Let's now turn to possible physical implementations of the honeycomb model. One thing that is important to emphasize is that neither the honeycomb geometry near the association of the spin-space component σ_i with a particular spatial direction i in the plane (or indeed the interpretation of $\boldsymbol{\sigma}$ in spatial terms at all) is in any way essential; the only things that are essential are the *topology* of the lattice points and the association of these different Pauli-like operators with the three distinguishable links. Unfortunately, it turns out that even with these relaxed conditions no naturally occurring system is likely to have a Hamiltonian particularly close to that of eqn. (29); for example, in the most obvious suggestion (graphene (honeycomb lattice) isotopically enriched with ^{13}C) the nuclear dipolar interaction has a quite different structure. Thus one turns to purpose-engineered systems. While there have been various proposals in the literature of the last few years, the most promising seem to be (a) optical lattices and (b) Josephson junction arrays. In the former case[7], it is proposed that one can create a honeycomb lattice of optical wells by using three coplanar lasers, and then induce a spin- and direction-dependent interaction between atoms in neighboring walls by appropriately polarized further lasers. In this way one would implement a "literal" version of the Kitaev honeycomb model (except that the 2D "spin" space actually refers to two different atomic hyperfine-Zeeman levels). The proposed (b) actually has two variants: in one of them[8] ("JJ *networks*") the superconducting wave function extends over the whole array and one manipulates its topological properties, while in the other[9] ("JJ *arrays*") one envisages individual Josephson systems (typically charge qubits) that are coupled through nonsuperconducting inductances, capacitances, etc.

Since it appears unlikely that any implementation, whether natural or purpose-engineered, can *exactly* replicate the Kitaev honeycomb Hamiltonian (29), the question naturally arises: how much are the conclusions drawn for that model against perturbations that spoil its simple structure, e.g. interactions of the form $-J_{xy}\sigma_i^x\sigma_j^y$? While one's instinct is that

[6] Kitaev's original argument on this point is based on the analogy with a $(p+ip)$ Fermi superfluid, which as we shall see in lecture 27 may be a little dangerous.

[7] L-M Duan et al., PRL **91**, 090402 (2003)

[8] L. B. Ioffe et al., Nature **415**, 503 (2002). The implementation proposed here is actually of the quantum dimer model, which is similar but not identical to the Kitaev honeycomb model.

[9] You et al., arXiv:0809.0051.

the basic results should not be affected provided that J_{xy} is much less than the original energy gap Δ, to my knowledge there has been little quantitative study of this issue in the literature to date.