Effects of interactions in a disordered system

The effects of the Coulomb and electron-phonon interaction in disordered systems, even where the disorder is "weak", is still a controversial subject. Here I will try to summarize the understanding that was obtained in the early and mid-80s, which appears to fit most of the pre-1997 data at least qualitatively. Although the electron-phonon interaction is believed to be very important in some disordered semiconductors,[1] it appears to be much less so in disordered metals, so I will concentrate here on the effects of the Coulomb interaction. It will always be assumed that we are in the "weak-disorder" limit $k_{\mathrm{F}}l \gg 1$.

The first point to make is that under this condition the static screening of the Coulomb interaction at long distances (small q) is likely to be essentially the same as in a 3D pure system. The reason is that the scale k_{FT}^{-1} of the screening cloud is so small (typically of the same order as ${k_{\mathrm{F}}}^{-1}$) that at these scales the system is "locally" effectively 3D[2], and moreover since the scale is $\ll l$ the system should not know about the disorder in this context. Thus we should be able to replace the long-wavelength ($q \ll k_{\mathrm{F}},\ l^{-1}$) spatial components of the static Coulomb interaction by a constant (the inverse "neutral" (3D) compressibility ${\chi_0}^{-1}$). Moreover, since the characteristic frequencies involved in the screening process are of order of the (3D) plasma frequency ω_p ($\sim$ a few eV), this effective interaction should be independent of frequency on the scales (at most of the order of the inverse elastic collision time τ^{-1}) which we will be interested in below. Thus, in the following, I will set $V_{\mathrm{eff}}(q,\omega) = \mathrm{const.}$ for the q and ω of interest.

In considering the effects of interactions in disordered systems, it is at first sight tempting to argue as follows: For a *noninteracting* degenerate system of electrons, the presence of disorder has little affect on the single-particle density of states (DOS) near the Fermi surface, and thus on properties such as the specific heat, compressibility and Pauli spin susceptibility. On the other hand, for a *pure* (non-disordered) system we know that (as long as no phase transition such as superconductivity sets in) the effects of the interactions can be handled by the Landau Fermi-liquid theory (or more precisely by the Landau-Silin generalization, which takes into account the long-range Coulomb force), and while they change the numerical coefficients (e.g. $m \to m^*$) do not affect the qualitative behavior (in particular, the DOS remains approximately constant). Thus, we should expect that also in a disordered interacting system the same qualitative result would hold (DOS $\sim$ constant, so $c_v \sim T$ etc.). This does indeed appear to have been the general assumption for nearly a quarter-century following Landau's original work. Interestingly, as first clearly appreciated by Altshuler and Aronov in 1979, it is *false*: while neither disorder nor interactions individually change the qualitative behavior, the combination does!

To see the reason, it is convenient to start by relating the decay rate Γ of a quasiparticle close to the Fermi energy to the density-density correlation function $\chi_{\rho\rho}(\mathbf{q}\omega)$ (hereafter simply $\chi(\mathbf{q}\omega)$). We will work in the basis of the *exact* single-electron energy eigenfunctions

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[1] See e.g., P. C. Taylor, J. Noncrystalline Solids, **352**, 839 (2006), section 5.

[2] This point may have to be re-examined in the context of Si MOSFETS, etc.

i in the absence of interaction (thus, in the absence of interaction Γ_i is trivially zero). The energy of the state i relative to the Fermi energy will be denoted ϵ_i and its Schrödinger wave function $\varphi_i(\mathbf{r})$. The single-particle DOS (of both spins) close to the Fermi will be denoted $dn/d\epsilon$; as noted, it is constant and approximately (i.e. to $\mathcal{O}({k_\mathrm{F}}^{-1})$) close to that of the pure metal. For simplicity I do not include the spin degree of freedom explicitly below, but it is straightforward to include it (at least in so far as the effective potential V_eff is spin-independent). Obviously the pure metal is a special case of the formalism to be developed below, with $\varphi_i(\mathbf{r}) = V^{-1/2} \exp i\mathbf{k}_i \cdot \mathbf{r}$ (V = volume of system).

Let's consider a state i, for definiteness with $\epsilon_i > 0$. In second-order perturbation theory the expression for its rate of decay Γ_i is

$$\Gamma_i = (2\pi/\hbar) \sum_{l,m,n} \left| \iint d\mathbf{r} d\mathbf{r}' \, \varphi_i^*(\mathbf{r}) \varphi_l(\mathbf{r}) V_\text{eff}(\mathbf{r}-\mathbf{r}') \varphi_m^*(\mathbf{r}') \varphi_n(\mathbf{r}') \right|^2 \\ \times f_m(1-f_l)(1-f_n)\delta(\epsilon_i + \epsilon_m - \epsilon_l - \epsilon_n) \tag{1}$$

where $f_m \equiv \big(\exp(\beta\epsilon_m)+1\big)^{-1}$ is the Fermi function. Intuitively, at $T = 0$ eqn. (1) describes a process in which the electron in state i knocks one in state m out of the Fermi sea (i.e. creates a hole in state m), the two outgoing electrons having energies $\epsilon_l, \epsilon_n > 0$. It turns out to be convenient to study, rather than Γ_i itself, the quantity

$$Q(\epsilon) = \sum_k \Gamma_k \theta(\epsilon_k)\theta(\epsilon - \epsilon_k) \tag{2}$$

i.e. $Q(\epsilon)$ is the sum of the decay rates of all electron states with energies less than ϵ. We now substitute (1) into (2) and use the identity

$$\delta(\epsilon_k + \epsilon_m - \epsilon_l - \epsilon_n) \equiv \int d\omega \, \delta\big(\omega - (\epsilon_k - \epsilon_l)\big)\delta\big(\omega - (\epsilon_n - \epsilon_m)\big) \tag{3}$$

thus obtaining

$$Q(\epsilon) = (2\pi/\hbar) \sum_{k,l,m,n} \int d\omega \left| \iint d\mathbf{r} \; d\mathbf{r}' \varphi_k^*(\mathbf{r}) \varphi_l(\mathbf{r}) V_\text{eff}(\mathbf{r}-\mathbf{r}') \varphi_m^*(\mathbf{r}') \varphi_n(\mathbf{r}') \right|^2 \\ \times f_m(1-f_l)(1-f_n)\,\theta(\epsilon_k)\theta(\epsilon - \epsilon_k)\delta\big(\omega - (\epsilon_k - \epsilon_l)\big)\delta\big(\omega - \epsilon_n - \epsilon_m\big) \tag{4}$$

The result (4) is exact within second-order perturbation theory.

At first sight, the expression (4) does not seem to be related to anything familiar. However, we now carry out a little trick.[3] Write $Q(\epsilon)$ formally in the form

$$Q(\epsilon) \equiv \sum_{kl} \theta(\epsilon_k)\;\theta(\epsilon - \epsilon_k)(1-f_l) \int d\omega \; \Phi_{kl}(\omega)\; \delta\big(\omega - (\epsilon_k - \epsilon_l)\big) \tag{5}$$

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[3] A very similar trick is routinely carried out in the discussion of the onset of superconductivity in disordered metals, see e.g. de Gennes, *Superconductivity of Metals and Alloys*, pp. 215–7.

The dependence of $\Phi_{kl}(\omega)$ on k and l is only through the product of wave functions $\varphi_k^*(\mathbf{r})\varphi_l(\mathbf{r})$ in the matrix element. The crucial observation, now, is that in the noninteracting picture that we are using as a basis, there is nothing in any way "special" about the states that happen to lie close to the Fermi energy, and thus we should expect that, at least when averaged over a nonzero but small energy range, the quantity $\Phi_{kl}(\omega)$ should not be a function of ϵ_k and ϵ_l separately, but *only of the difference* $\epsilon_k - \epsilon_l$, and moreover is symmetric under the exchange $k \rightleftarrows l$., Thus, performing the sum over l in (5), we can write

$$Q(\epsilon) = \left(\frac{dn}{d\epsilon}\right)^2 \int_0^\epsilon d\epsilon' \int_{-\infty}^{\infty} d\omega \big(1 - f(\epsilon' - \omega)\big)\Phi(\omega) \tag{6}$$

Let us now compare expression (6) with the expression

$$\begin{aligned}\tilde{Q}(\epsilon) &\equiv \int_0^\epsilon d\omega \sum_{kl} f_l(1-f_k)\, \delta\big(\omega - (\epsilon_k - \epsilon_l)\big)\, \Phi_{kl}(\omega) \\ &= \left(\frac{dn}{d\epsilon}\right)^2 \int_0^\epsilon d\omega \int_{-\infty}^{\infty} d\epsilon'\, f(\epsilon' - \omega)\big(1 - f(\epsilon')\big)\Phi(\omega)\end{aligned} \tag{7}$$

Consider the limit $T \to 0$, so that $1 - f(\epsilon) \to \theta(\epsilon)$. Then the expression (6) and (7) reduce respectively to

$$Q(\epsilon) = \left(\frac{dn}{d\epsilon}\right)^2 \int_0^\epsilon (\epsilon - \omega)\Phi(\omega)d\omega + \epsilon I, \quad I \equiv \int_{-\infty}^0 \Phi(\omega)d\omega \tag{8}$$

$$\tilde{Q}(\epsilon) = \left(\frac{dn}{d\epsilon}\right)^2 \int_0^\epsilon \omega\Phi(\omega)d\omega \tag{9}$$

From the consideration of the second δ-function, etc., in $\Phi(\omega)$ we can argue that at $T = 0$ $\Phi(\omega) \equiv 0$ for $\omega < 0$, hence $I = 0$. Thus $Q(\epsilon)$ and $\tilde{Q}(\epsilon)$ differ only by the factors of $\epsilon - \omega$ and ω, respectively, in their intergrands; for any simple power-law form of $\Phi(\omega)$ this difference will contribute a calculable numerical factor $\alpha \equiv Q(\epsilon)/\tilde{Q}(\epsilon)$ which is independent of ϵ.

Thus, we can write (at $T = 0$)[4]

$$\begin{aligned}Q(\epsilon) = \alpha(2\pi/\hbar) \int_0^\epsilon d\omega \sum_{klmn} &\left| \iint d\mathbf{r}d\mathbf{r}'\, \varphi_k^*(\mathbf{r})\varphi_l(\mathbf{r})V_{\text{eff}}(\mathbf{r} - \mathbf{r}')\varphi_m^*(\mathbf{r}')\varphi_n(\mathbf{r}') \right|^2 \\ &\times \big\{ f_l(1-f_k)f_m(1-f_n)\delta\big(\omega - (\epsilon_k - \epsilon_l)\big)\delta\big(\omega - (\epsilon_m - \epsilon_n)\big) \big\}\end{aligned} \tag{10}$$

Now at last we are in a position to relate $Q(\epsilon)$ to the density-density response function $\chi(\mathbf{r}\mathbf{r}' : \omega) \equiv \delta\rho(\mathbf{r} : \omega)/\delta U(\mathbf{r}' : \omega)$. We recall that in the absence of interactions

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[4] I believe it should be possible to generalize the argument to nonzero T, but have not to date done so.

$$\operatorname{Im}\chi(\mathbf{r},\mathbf{r}':\omega)=\sum_{ij} f_i(1-f_j)\ \varphi_i^*(\mathbf{r})\ \varphi_j(\mathbf{r})\ \varphi_j^*(\mathbf{r}')\ \varphi_i(\mathbf{r}')\ \delta\big(\omega-(\epsilon_j-\epsilon_i)\big) \tag{11}$$

and so, expanding the squared matrix elements in (10) explicitly,

$$\begin{aligned}Q(\epsilon)=\alpha\,(2\pi/\hbar)\int_0^\epsilon d\omega\iint d\mathbf{r}_1 d\mathbf{r}_2' d\mathbf{r}_2 d\mathbf{r}_2\ \operatorname{Im}\chi(\mathbf{r}_1,\mathbf{r}_1':\omega)\operatorname{Im}\chi(\mathbf{r}_2,\mathbf{r}_2':\omega)\\ \times V_{\text{eff}}(\mathbf{r}_1-\mathbf{r}_2)V_{\text{eff}}(\mathbf{r}_1'-\mathbf{r}_2')\end{aligned} \tag{12}$$

This expression is still a bit messy. To simplify it, we use the fact that when averaged over the COM variable $\chi(\mathbf{r}_1,\mathbf{r}_1')$ is a function only of the difference variable $(\mathbf{r}_1-\mathbf{r}_1')$. Then taking Fourier transforms we find

$$Q(\epsilon)=\alpha\,(2\pi/\hbar)\int_0^\epsilon d\omega\int d^d\mathbf{q}\ |V_{\text{eff}}(\mathbf{q})|^2\ \{\operatorname{Im}\chi(\mathbf{q}\omega)\}^2 \tag{13}$$

Since $V_{\text{eff}}(\mathbf{q})$ can be taken to be some constant V_0, we finally obtain (at $T=0$) for $\Gamma(\epsilon)\equiv(dn/d\epsilon)^{-1}\partial Q(\epsilon)/\partial\epsilon$ the simple expression

$$\boxed{\Gamma(\epsilon)=\alpha\,(2\pi/\hbar)V_0^2(dn/d\epsilon)^{-1}\int d^d\mathbf{q}\ |\operatorname{Im}\,\chi(\mathbf{q},\epsilon)|^2} \tag{14}$$

Although it was hard work to get to eqn. (14), the payoff is worth it since (a) it is valid independently of the degree of disorder and the dimensionality d of the system (through the numerical constant α will depend on these), and (b) it is often possible to use simple phenomenological arguments to determine the form of $\operatorname{Im}\chi(q,\epsilon)$ in the "interesting" regime. Let's start with the pure case, which should be well described by Fermi-liquid theory, or at least qualitatively by the Sommerfeld free-electron-gas model. For this model we have for $\epsilon\ll\epsilon_{\text{F}}$ and $q\ll q_{\text{F}}$ the simple result (in 3D)

$$\operatorname{Im}\chi(\mathbf{q},\epsilon)=(dn/d\epsilon)(\epsilon/qv_{\text{F}})\ \theta(qv_{\text{F}}-\epsilon) \tag{15}$$

The principal contribution to the integral over q in (14) then comes from *large* q $(qv_{\text{F}}\gg\epsilon)$ (of course, for $q\sim q_{\text{F}}$ expression (15) fails and we have to use a more accurate expression, but by that time the approximation $V_{\text{eff}}(\mathbf{q})=V_0$ has already failed). The important point is that once the explicit factor of ϵ^2 is taken out the result of the integration is only negligibly dependent on ϵ. Thus we obtain the familiar result

$$\Gamma_{\text{FL}}^{(3\text{D})}(\epsilon)=\text{const.}\ \epsilon^2 \tag{16}$$

(which could of course have been derived more quickly by the standard arguments based on the Pauli principle).

For a pure 2D system the situation is less obvious. Because of the square-root divergence of the noninteracting response function $\chi_0(\mathbf{q}\omega)$ for $\omega \to qv_{\mathrm{F}}$ [Problem], evaluation of (15) using this leads to the result

$$\Gamma^{\mathrm{2D}}_{\mathrm{FL}}(\epsilon) = \text{const. } \epsilon^2 \ln(\epsilon_F/\epsilon) \tag{16a}$$

However, it is not immediately clear whether replacing $\chi(\mathbf{q}\omega)$ with the RPA removes the singularity. According to what as far as I know are the most recent results,[5] it does not, i.e. (16a) is the correct result. In any case the point is somewhat moot since, as we shall see, any nonzero disorder always renders (16a) incorrect in the limit $\epsilon \to 0$.

We now turn to the case of prime interest in the present context, namely the effects of electron-electron interactions in a disordered system. We should assume that the energy ϵ of interest is $\ll 1/\tau$ where τ is the *elastic* scattering time (remember that even under these conditions the standard Pauli-principle argument still gives $\Gamma(\epsilon) \sim \epsilon^2$). Suppose also that we concentrate for the moment on the regime of small q, $ql \ll 1$ where $l = v_{\mathrm{F}}\tau$ is the mfp against elastic scattering. Then the propogation of the density is given by the standard diffusion equation

$$\frac{\partial \rho(\mathbf{r}t)}{\partial t} = D\nabla^2 \rho(\mathbf{r}t) \tag{17}$$

where D $(= \sigma/\chi_0)$ is the diffusion coefficient, and correspondingly the density-density correlation function has the standard ("dirty-hydrodynamic-limit") form

$$\chi(\mathbf{q}\omega) = \chi_0 \frac{D\mathbf{q}^2}{-i\omega + D\mathbf{q}^2} \tag{18}$$

Thus from (14) we find that the small-q contribution to $\Gamma(\epsilon)$ (call it $\Gamma_{\mathrm{SR}}(\epsilon)$) is

$$\Gamma_{\mathrm{SR}}(\epsilon) = \text{const. } \epsilon^2 \int_0^{q_m} \frac{q^{d-1}(D\mathbf{q}^2)^2}{(\epsilon^2 + D^2\mathbf{q}^4)^2}\, d\mathbf{q} \tag{19}$$

where q_m is a phenomenological cutoff, of order l^{-1}. Without evaluating the integral explicitly,[6] it is clear that (for $d < 4$) it is of order $\epsilon^{-2+d/2}$ and hence we find

$$\boxed{\Gamma_{\mathrm{SR}}(\epsilon) = \text{const. } \epsilon^{d/2}} \tag{20}$$

Since in $d \leq 3$ dimensions this power low is lower than the ϵ^2 that would come from large q, it is clear that in the limit $\epsilon \to 0$ eqn. (20) is the exact asymptotic form of $\Gamma(\epsilon)$.

Now let us calculate the renormalization of the single-electron density of states (DOS) in this approximation (lowest-order perturbation theory in the interaction). To do this, it

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[5] Metzner et al., Adv. Phys., **47**, 317 (1998).

[6] For $d = 2$ the integral is trivial and equal to $(\pi/8)(\epsilon/D)$.

is necessary to calculate the shift in energy due to the virtual processes that correspond to the real processes that are treated in eqn. (1): formally,

$$\Gamma_n = \sum_i |\langle i|V|n\rangle|^2 \delta(\epsilon_i - \epsilon_n) \tag{21a}$$

$$\Delta\epsilon_n = \sum_i \frac{|\langle i|V|n\rangle|^2}{\epsilon_n - \epsilon_i} \tag{21b}$$

and so

$$\Delta\epsilon_n = \int \frac{\Gamma_n(\epsilon')}{\epsilon_n - \epsilon'} d\epsilon' \tag{22}$$

where $\Gamma_n(\epsilon')$ means the value of Γ_n which would be calculated by substituting $\epsilon_n = \epsilon'$ in eqn. (21a). In the standard textbook field—theoretic technique, the equation corresponding to (22) (which is actually valid beyond second-order perturbation theory) relates the real part $\Sigma(\epsilon)$ of the single-electron self-energy to its imaginary part $\Gamma(\epsilon)$:

$$\Sigma(\epsilon) = \int_{-\infty}^{\infty} \frac{d\epsilon'}{\epsilon - \epsilon'} \Gamma(\epsilon') \tag{23}$$

The only slightly tricky point here is that the integral runs over negative[7] as well as positive ϵ', so that one needs to know also $\Gamma(\epsilon)$ for the hole states ($\epsilon < 0$); however, we can see immediately from the symmetry around the Fermi surface that this must be a reflection of the positive-energy behavior, i.e. $\Gamma(\epsilon) \sim \text{const} \mid \epsilon \mid^{d/2}$ for $\epsilon < 0$.

If we substitute the 3D Fermi liquid form of $\Gamma(\epsilon)$, eqn. (16), in (23), we find that the lowest-order[8] correction $\Sigma(\epsilon)$ to the original energy ϵ is of order ϵ^2, and the correction is thus negligible in the limit $\epsilon \to 0$. For the 2D Fermi liquid the situation is not entirely clear, but it seems unlikely that the behavior is worse than $\epsilon^2 \ln \epsilon$, which still vanishes by comparison with ϵ in the limit $\epsilon \to 0$. Turning now to the disordered case, we see that for $d = 3$ eqn. (23) leads to $\Sigma(\epsilon) \sim \epsilon^{3/2}$, still negligible compared to ϵ. However, for $d = 2$ things change qualitatively: the lowest-order term in ϵ in $\Sigma(\epsilon)$ is

$$\Sigma(\epsilon) \sim -\text{const.}\, \epsilon \ln(\epsilon_c/\epsilon) \tag{24}$$

where ϵ_c is an upper cutoff energy in the integral (23), which we can take to be of the order of the inverse of the elastic scattering time τ (since for $\epsilon \gtrsim 1/\tau$ the hydrodynamic (diffusion) approximation (18) for $\chi(\mathbf{q}\omega)$ fails). The novel feature is that now the (negative) correction to the original energy ϵ of the state n now exceeds ϵ itself in the limit $\epsilon \to 0$. This indicates a fundamental breakdown of Fermi-liquid theory, for a dirty 2D metal. In fact, it

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[7] This is not entirely obvious intuitively and needs detailed justification: see e.g. AGD.

[8] There is in fact a nonanalytic connection of order $\epsilon^3 \ln \epsilon$, but this would need a more accurate treatment of $\Gamma(\epsilon)$ to discuss it.

is clear that we cannot take the above results (based on lowest-order perturbation theory) seriously when $\epsilon \to 0$. To determine the order of magnitude of ϵ at which perturbation theory fails, we need to compare ϵ with $\Sigma(\epsilon)$ as calculated in (24). Now the constant in eqn. (20) (which within a factor of order 1 is the same as that in (24)) may be seen from eqn. (14) and the fact that $\chi_0 \sim (dn/d\epsilon)$ (the exact equality holding for the Sommerfeld-gas model) to be of order $V_0^2(dn/d\epsilon)/D$, which generally speaking should be of order[9] $(k_{\mathrm{F}}l)^{-1}$. Hence, the value ϵ_c of ϵ for which perturbation theory in the interaction should fail should be given (to a small order of magnitude) by

$$\ln\ \epsilon_c\tau/\hbar \sim -1/k_{\mathrm{F}}l, \tag{25}$$

i.e.,

$$\epsilon_c \sim \frac{\hbar}{\tau}\exp -1/k_{\mathrm{F}}l. \tag{26}$$

For any reasonably clean metal the value of ϵ_c is so small as to correspond to unattainably low temperatures, which explains why these effects are not routinely seen. However, in dirty metals, where $k_{\mathrm{F}}l$ is not much larger than 1 they shall be observable (cf. below) and in fact it may not be so difficult to make the point where perturbation theory breaks down.

The most obvious effect of eqn. (24) should be the renormalization of the single-electron DOS. Assuming that we are in the "perturbation" limit $\Sigma(\epsilon) \ll \epsilon$, we have $\big(dn/d\epsilon \to \nu(\epsilon)\big)$

$$\frac{\nu(\epsilon)}{\nu_0(\epsilon)} = \frac{1}{1+\partial\Sigma/\partial\epsilon} \cong 1 - \partial\Sigma/\partial\epsilon \tag{27}$$

i.e. for the correction $\delta\nu(\epsilon)$ to the free-gas value $\nu_0(\epsilon)$

$$\delta\nu(\epsilon) \sim \mathrm{const.}\ln(\epsilon_c/\epsilon). \tag{28}$$

At finite temperatures the generalization is probably best done by the standard techniques of graphical perturbation theory, but the result is what one might perhaps guess: for $T \gg \epsilon$ the ϵ in (28) is simply replaced by T, i.e.

$$\delta\nu(\epsilon) \sim \mathrm{const.}\ln(\epsilon_c/T). \tag{29}$$

while for $T \ll \epsilon$ the zero-temperature result (28) is preserved.

In principle the logarithmic behavior of the DOS should lead to a similar behavior of the specific heat of a 2D sample at sufficiently low temperatures, i.e.

$$\delta C_v(T) \sim T\ln(\epsilon_c/T). \tag{30}$$

but this turns out to be very difficult to measure. There are, however, tunnelling measurements[10] that seem to confirm more directly the energy- and temperature-dependence of the DOS.

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[9] We use $V_0^2 \sim (dn/d\epsilon)^{-1} \sim \hbar^2/m$ in 2D, and $D \sim v_F l$.

[10] Imry and Ovadyahu, PRL **49**, 841 (1982)

However, as noted in lecture 3, the easiest properties of 2D and quasi-2D systems to measure are the transport coefficients, and in particular the dc conductivity σ. One might expect that in view of the $\ln T$ dependence of the DOS, this would have a similar behavior, but the sign of the effect is not a priori obvious – the effective number of carriers is in some sense increased, but so is the probability of $e-e$ collisions, which unlike (N-processes) in the pure case do not conserve momentum. In fact it seems impossible to proceed without a proper graphical field theory calculation. It is relatively straightforward to carry out such a calculation to second order in the interaction; note that the graphs which are summed in this calculation do *not* correspond to those necessary to obtain the WL connection in the noninteracting system, so that this approximation treats two effects as mutually independent. The result is

$$1D: \qquad \delta\sigma_{e-e}(T) = -\frac{1}{A}\,\frac{e^2}{2\pi\hbar}(4-\frac{3}{2}\tilde{F}_\sigma)(D/2T)^{1/2} \sim T^{-1/2} \tag{31}$$

$$2D: \qquad \delta\sigma_{e-e}(T) = -\frac{e^2}{4\pi^2\hbar}(2-\frac{3}{2}\tilde{F}_\sigma)\ln(1/T\tau) \tag{32}$$

where the dimensionless quantity $\tilde{F}_\sigma$ is a rather complicated average of the screened Coulomb matrix elements over the Fermi surface (see Lee and Ramakrishnan, op. cit., p. 308), which while always positive is not a priori large or small compared to 1.

We recall, now, that for the noninteracting system in the absence of magnetic fields and spin-orbit scattering, the WL connection to the conductivity is (lecture 5)

$$1D: \qquad \delta\sigma_{WL}(T) = -\frac{e^2}{\pi\hbar}L_\phi(T) \sim T^{-p/2} \tag{33}$$

$$2D: \qquad \delta\sigma_{WL}(T) = -\frac{e^2}{\pi^2\hbar}\ln L_\phi(T)/l = (\text{const.}+) - \frac{e^2 p}{2\pi^2\hbar}\ln(1/T\tau) \tag{34}$$

where $L_\phi(T) \equiv \sqrt{D\tau_\phi(T)}$ is the "phase-breaking length" and the corresponding "phase-breaking rate" τ_ϕ^{-1} is assumed to depend on T as T^p.

How can one tell, experimentally, if the temperature-dependence one is seeing is due to disorder or to $e-e$ interactions or both?

For a 1D system, the predicted temperature-dependence is usually different (since usually $p>1$), and in fact we see that in the limit $T \to 0$ the "weak-localization" effect will generally dominate the "interaction" one. In 2D we cannot use a similar argument, since both effects are proportional to $-\ln T$. However, in this case the magnetic field-dependence is a useful diagnostic. The WL effect is sensitive only to the component of the field $\perp$ to the plane, and the temperature-dependence of $\delta\sigma(T)$ is predicted to saturate when the phase-breaking length $L_\phi(T)$ becomes comparable to the magnetic length L_M $\big(\equiv (\hbar/eB_\perp)^{1/2}\big)$. As regards the interaction effect, it turns out that it is insensitive to the orbital coupling; it does have a field-dependence (corresponding to *positive* magnetoresistance) due to the Zeeman (spin) coupling, but this is independent of the field direction[11] and has a characteristic scale given by $g\mu_\mathrm{B}B \sim k_\mathrm{B}T$. At low temperatures the corresponding "crossover" field B_z is usually much greater than that (B_orb) for the orbital effect on WL, so that by choosing a field B such that $B_{orb} \ll B \ll B_z$ one can isolate the part of the $\ln T$ temperature-dependence which is due to the interaction effect.

Putting these results together with the standard WL results derived in lectures 6-8, we reach the following general conclusions:[12]

(1) If $\tau_\phi(T)$ is measured, e.g. by measuring the crossover field in the magnetoresistance as a $f(T)$, it should tend to ∞ as $T \to 0$ (as some negative power of T) except for the effect of static magnetic impurities [in particular, $e^- - e^-$ interactions should give $\tau_\phi^{-1} \sim T$ in 2D and $T^{2/3}$ in 1D: see LR section III.2]

(2) If we simply add the contributions to the correction $\Delta\sigma(T)$ to the Boltzmann conductivity in 2D from WL and from interactions (as seems reasonable when both are small) we find that apart from an uninteresting (and unmeasurable!) constant the total correction is given by the expression

$$\Delta\sigma(T) = -(e^2/\pi\hbar)(\alpha p + 1 - \frac{3}{4}\tilde{F}_\sigma)\ln(1/T\tau_\phi) \tag{35}$$

where p is the power of T with which τ_ϕ^{-1} tends to zero as $T \to 0$, and α is a constant of order 1. Although values of the "screening function" $\tilde{F}_\sigma$ up to 3.5 have been measured in Si MOSFETs, to make the factor in (35) negative would require $\alpha p < 1.6$, which seems unlikely though not impossible. Unless this happens, or the situation changes qualitatively when $\Delta\sigma(T)$ becomes comparable to σ_0, one would expect that a *normal-metallic state*[13] *cannot exist in 2D as* $T \to 0$. This was indeed the accepted wisdom until around 1995.

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[11] Unless the spin-orbit interaction is so strong as to constrain the direction of the spins.

[12] It is essential, here, to remember that the "anomalous" effects of SO scattering discussed in Lecture 8 require that the scattering be effectively *3D*. In Si MOSFET's or GaAs heterostructures, it is effectively 2D and the SO scattering simply adds to the dephasing terms.

[13] A superconducting state may exist, subject to the usual KT considerations, since the effect of "localization" on Cooper pairs is qualitatively different from that on single electrons.