Experimental tests of the BKT theory

Since it may be a bit difficult to see the wood for the trees, let's start by reviewing the main predictions of the BKT theory which might be put to experimental test. First, the qualitative picture: The energy of a vortex in a 2D film of dimension R diverges as $\ln R/r_0$, so at low temperatures there are no free vortices. Vortex-antivortex pairs, which have a finite energy, can occur, but are unable to nucleate decay of the dc supercurrent in the limit $\mathbf{v}_s \to 0$, so the system is superfluid. At a characteristic temperature T_{KT} the vortex-antivortex pairs become unbound (their radius tends to ∞) so for $T > T_{\mathrm{KT}}$ we have many free vortices, which can move across the supercurrent and destroy it; thus the system is normal (non-superfluid). A complication is that even below T_{KT} the bound pairs can contribute to decay of the supercurrent for either nonzero $\mathbf{v}_s$ or nonzero frequency ω.

Let's try to be a bit more quantitative. First, as to the static properties: For $T < T_{\mathrm{KT}}$ the correlation of the order parameter,

$$C(|\mathbf{r} - \mathbf{r}'|) \equiv \langle \psi^*(\mathbf{r})\psi(\mathbf{r}')\rangle \tag{1}$$

falls off algebraically in the limit $|\mathbf{r} - \mathbf{r}'| \to \infty$:

$$C(\mathbf{r}) = \text{const.}\, r^{-\eta(T)} \tag{2}$$

where $\eta(T)$ is proportional to T and tends to the value 1/4 as $T \to T_{\mathrm{KT}}$ from below. This behavior is due to the effect of *small* fluctuations around equilibrium: for r$\gg \xi_-$ vortices do not contribute. The important physical effect in this regime is the screening of the interaction of a given vortex-antivortex pair at points $\mathbf{r}$, $\mathbf{r}'$ by the polarization of other vortex-antivortex pairs lying between them; we can define an "effective" superfluid density $\rho_s(|\mathbf{r}-\mathbf{r}'|)$ as proportional to the screened interaction, and in the electromagnetic analogy this is then proportional to $1/\epsilon(|\mathbf{r} - \mathbf{r}'|)$ ($\epsilon(|\mathbf{r} - \mathbf{r}'|) \to \epsilon(r)$ from now on). The effective superfluid density starts off, at a scale $\sim$ the vortex radius r_0, at the GL value $\rho_s^0(T)$; the effect of screening is to renormalize it downwards, so that it approaches the experimentally measured value $\rho_s(T)$ at $r \to \infty$. The characteristic scale at which the crossover from $\rho_s^0(T)$ to $\rho_s(T)$ takes place, ξ_-, has the temperature-dependence

$$\xi_-(T) =\sim r_0 \exp(b|t|)^{-1/2}, \quad t \equiv T - T_{\mathrm{KT}}\ (<0) \tag{3}$$

(where b is a nonuniversal constant), and thus diverges very fast as $T \to T_{\mathrm{KT}}$ from below. In this limit the experimentally measured (dc) superfluid density $\rho_s(T)$ is predicted to satisfy the *universal* relation

$$\rho_s(T \to T_{\mathrm{KT}}^{(-)}) = \frac{2}{\pi}\left(\frac{m}{\hbar}\right)^2 k_{\mathrm{B}} T_{\mathrm{KT}} \tag{4}$$

For $T > T_{\mathrm{KT}}$ the effective superfluid density again starts off, at scale r_0, at the GL value (which for T less than the mean-field transition temperature at which $\alpha(T) \to 0$ is still nonzero), but now scales to 0 as $r \to \infty$, the transition taking place over a distance of order $\xi_+(T)$ given by

$$\xi_+(T) \sim r_0 \, \exp(b't)^{-1/2} \tag{5}$$

As a result (Problem) the order parameter correlation $C(\mathbf{r}) \equiv \langle\psi^*(0)\psi(\mathbf{r})\rangle$ falls off algebraically for $r \ll \xi_+(T)$, but at longer length scales falls off *exponentially*:

$$C(\mathbf{r}) \sim \exp -cr/\xi_+(T) \quad r \to \infty \; (c \sim 1) \tag{6}$$

Turning to the consequences for the dynamics, we see that for $T > T_{\mathrm{KT}}$ the situation is straightforward: free vortices exist, and for arbitrary small $\mathbf{v}_s$ can move across the supercurrent and annihilate it. Since the Magnus force is proportional to $\mathbf{v}_s$, this gives rise to a linear damping. However, the density of *unpaired* vortices is proportional to $\xi_+^{-2}(T)$, so one predicts that the linear friction coefficient (or in the case of a charged system the linear resistance R) should satisfy the relation

$$R(T) \sim \xi_+^{-2} \sim \exp -2b't^{-1/2} \tag{7}$$

For $T < T_{\mathrm{KT}}$ the situation is more complicated and needs to be analyzed as in lecture 11: the upshot is that $(-d\mathbf{v}_s/dt) \propto \mathbf{v}_s^3$ in the dc case and for the ac case the effective value of the "dielectric constant" $\epsilon(\omega)$ is complex and given by eqns. (11.28); thus it can be calculated from an explicit solution of the Kosterlitz equations (11.14).

In reviewing experimental tests done to date of the predictions of BKT theory, it has to be borne in mind that any particular experimental system will generally only allow tests of a subset of the above predictions; indeed, as far as I know no system currently exists which will permit tests of *all* the static and dynamic behavior predicted. As we shall see, He films allow measurements only of the finite-frequency "dielectric constant" $\epsilon(\omega)$ (or equivalently the finite-frequency superfluid density); superconducting films, and also arrays of Josephson junctions, have permitted us to verify the predictions concerning the dc behavior of the resistivity, including the nonlinear aspects; while to obtain information on the correlations of the order parameter itself one needs to use alkali-gas Bose condensates. As we shall see each of those systems involves some complications with respect to the pristine BKT model.

One particular complication of which one should be aware is the possible effect of any normal component which may be present. That such normal component, even if present at relatively low level, may have highly nontrivial effects, possibly not accounted for in the "pure" BKT theory, is suggested by the puzzling data obtained on ^{3}He-^{4}He mixtures (see below); in these mixtures the "normal fraction" as defined in the standard 2-fluid model may be somewhat greater than the actual concentration of ^{3}He by number, but seems very unlikely to be more than $\sim 25\%$: In general, we would expect to be able to neglect the normal component if T_{KT} is less than say 0.5 of the mean-field (3D) transition transition temperature T_{co}; this condition turns out to be well fulfilled for Josephson junction arrays and Bose alkali gases, but is more marginal for (pure) ^{4}He films and probably not at all well fulfilled for thin metallic films.

The most systematic attempt to test the dynamic KT theory is that of Bishop and Reppy.[1] on ^{4}He films. They used the Andronikashvili technique, with a torsional oscillator of rotational frequency 2.5 KHz and a Q of $> 10^5$; the claimed to be able to resolve the effective moment of inertia (which does not include that of the "superfluid fraction" of the helium film, see below) to 5 parts in 10^9. The oscillator was wrapped with Mylar which gave a very large surface ($\sim 0.2\,\mathrm{m}^2$) for absorption of helium. The (average) coverage by the helium ranged from zero to $\sim 36\,\mu\mathrm{mol/m}^2$, corresponding roughly to 0 to 2 atomic layers.[2] For this type of experiment, it is straightforward to show that the shift ΔP in the oscillator period relative to the "normal" state where all the helium moves with the substrate, and the dissipation Q^{-1}, are related to the complex "dielectric constant" we calculated by

$$\Delta P/P = \frac{1}{2}(A/M)\rho_s(T_c^-)\mathrm{Re}\,\epsilon^{-1}(\omega,T) \tag{8}$$

$$Q^{-1} = (A/M)\rho_s(T_c^-)\mathrm{Im}\left[-\epsilon^{-1}(\omega,T)\right] \tag{9}$$

where M is the (unloaded) oscillator mass, A the area of coverage and $\rho_s(T_c^-)$ is the "macroscopic" value of the superfluid mass per unit area on the low side of the transition. In formulae (8,9) the "dielectric constant" (renormalization of the superfluid density) $\epsilon(\omega,T)$ is evaluated at the (fixed) frequency ω of the oscillator and at temperature T; as indicated in lecture 11, it can be related to the *dc* value of $\epsilon(\mathbf{r})$ at that temperature. Of course, in real life Q^{-1} is likely to have a background contribution (due to dissipation in the normal component, friction in the bearings etc.).

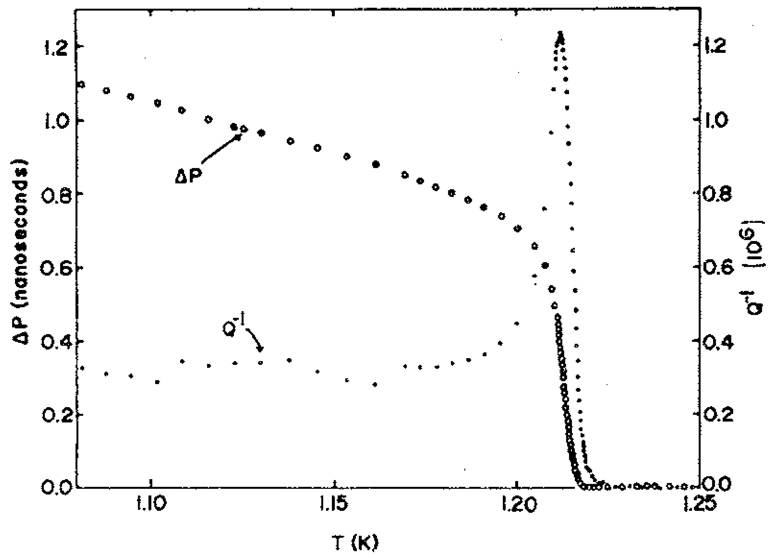
BR found that for coverages of pure ^{4}He less than $\sim 25\,\mu\mathrm{mol/m}^2$ (rather more than 1 monolayer) there was no temperature at which $\Delta P/P$ underwent any appreciable change, indicating that such films do not become "superfluid" down to $T = 0$ (presumably because they are "solid"). For higher coverages they found a relatively abrupt change (rise) in $\Delta P/P$ at a temperature which scaled linearly with the "excess" coverage with a maximum value of about 1.25 K at the maximum coverage of $36\,\mu\mathrm{mol/m}^2$; note that this is very considerably below the T_c of bulk liquid He ($\sim 2.17\,\mathrm{K}$), so that it is probably not a bad approximation to take the "mean-field" ρ_s (ρ_s^0) to be essentially given by its zero-T value ρ and thus to be independent of temperature. The *naive* estimate of T_{KT} (which makes it proportional to ρ_s^0 not ρ_s) then gives a linear dependence of T_{KT} on the areal coverage, with a predicted slope of $\sim 3.3 \times 10^{-9}\,\mathrm{gm/cm^2K}$; the experimental value is $\sim 3.5 \times 10^{-9}\,\mathrm{gm/cm^2K}$.

BR then studied the behavior of $\Delta P/P$ and Q^{-1} near T_{KT} in detail as a function of T, and fitted it to formulae derived from the dynamic KT theory (see above).

Let's ask what we would qualitatively expect. Recall that according to the results of AHNS quoted in lecture 11, below T_{KT} there is associated with frequency ω a characteristic length $r_\omega \approx (14D/\omega)^{1/2}$ where D is the vortex diffusion coefficient, and the real

[1] Phys. Rev. B **22**, 5171 (1980)

[2] I assume a coverage of 10^{15} He atoms/cm^2.



and imaginary parts of $\epsilon(\omega)$ are given up to numerical constants of order unity by

$$\mathrm{Re}\,\epsilon(\omega) = \tilde{\epsilon}(r_\omega, T) \tag{10}$$

$$\mathrm{Im}\,\epsilon(\omega) = \left(r\frac{d\tilde{\epsilon}}{dr}(r,T) \right)_{r_\omega} \tag{11}$$

where $\tilde{\epsilon}(r,T)$ is the *static* dielectric constant at scale r. We moreover recall that the length scale at which $\tilde{\epsilon}(r,T)$ changes (increases) appreciably is $\xi_-(T)$, where $\xi_-(T)$ increases fast as T approaches T_{KT} from below. Hence we conclude that for given ω there is a characteristic temperature T_0 such that

$$\xi_-(T_0) \sim (14D/\omega)^{1/2} \tag{12}$$

The real part of $\epsilon^{-1}(\omega)$ should be approximately constant for $T \ll T_0$, and for $T \sim T_0$ should drop smoothly to zero, while the imaginary part should, like $\mathrm{Im}\,\epsilon$ itself, be strongly peaked, as a function of T, around T_0. The data indeed show just this behavior. In fact, BR were able to get a fairly impressive fit for a film $\sim 35\mu\mathrm{mol/m}^2$ using the fit parameters

$$\begin{aligned} T_{\mathrm{KT}} = 1.2043\mathrm{K}, \quad \rho_s(T_c^-)A/M = 3.4 \times 10^{-6}, \quad F = 1.2 \\ \epsilon' = 0.07, \quad \ln(14D/r_0^2\omega) = 12 \end{aligned} \tag{13}$$

(here F is the fitting overall constant multiplying the free-vortex contribution and ϵ' is the "background" contribution to the dissipation). With these parameters the fits to the approximated theory ($x + 2 \approx 2$) are fairly good (see BR's fig. 10) and the fits to the unapproximated one even better (fig. 12).

There have been a number of subsequent experiments on pure ^{4}He films, most of which have shown reasonable agreement with the theoretical predictions for $\epsilon(\omega, T)$. In particular, a torsional-oscillator experiment by Bowley et al.[3] systematically studied films of ^{4}He on hydrogen-deuteride plated graphite, with thicknesses d ranging from 1 to 3 atomic layers. They found that when $\mathrm{Re}\,\epsilon^{-1}(T)$ is plotted against $\mathrm{Im}\,\epsilon^{-1}(T)$ (of course at fixed ω) a high degree of "collapse" is obtained (i.e. the curves for different d fall nearly on top of one another), although they do not in fact fit the theoretically expected curve all that well. In another recent experiment, Hieda et al.[4] were able, by exciting overtones of their microbalance, to measure the *frequency-dependence* of $\epsilon(\omega)$ at constant T, and found that the theoretical prediction is well reproduced.

Thus, so far the data seems to confirm the theory excellently. However, there are a couple of problems:

(a) The interpretation given by BR of their data relies on the film being completely uniform: if that is not so, then presumably T_{KT} and hence ϵ would be position-dependent and what one would measure is some kind of complicated average. In fact, there seems no obvious reason to believe that Mylar (the substrate in this experiment) is particular smooth, which suggests that the agreement with theory may be partly serendipitous. However, it seems less likely that this objection applies to the HD-plated graphite substrate used by Bowley et al.

(b) All the results so far quoted relate to films of pure ^{4}He. When ^{3}He is added (up to $\sim$ 10% concentration, the maximum stable one) the result change dramatically; while as far as I know all experiments to date have seen an anomaly identifiable as the KT transition, several of them[5] (including those of BR) also see a second anomaly at a somewhat lower temperature (and its features are not always reproducible between experiments). The nature of this second anomaly is at the time of writing quite obscure; as mentioned earlier, its occurrence might suggest that the role of an appreciable normal fraction may not be as innocent as much of the literature has assumed.

I now turn more briefly to the superconducting case. As regards the symmetry of the order parameter, this is identical to the case of a neutral superfluid, since in each case the order parameter is a complex scalar object and a gradient of its phase is associated with a supercurrent; the difference lies only in the microscopic interpretation of the objects described by it (Cooper pairs as distinct from single He atoms). However, a more important difference appears to lie in the fact that the order parameter of a superconductor interacts strongly with the electromagnetic field, resulting in the Meissner effect: as a result, in a bulk 3D superconductor the supercurrent due to a vortex falls off *exponentially* at distances $\gtrsim \lambda_{\mathrm{L}}$ (the London penetration depth); as a result, the total energy of a vortex is finite, the vortex-antivortex interaction is exponentially small for $r \gtrsim \lambda_{\mathrm{L}}$ and none of the considerations of KT would appear to apply. Thus, in their original paper KT remark that they would not expect their analysis to apply to superconducting films.

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[3] JLTP **113**, 399 (1998).
[4] J. Phys. Soc. Jpn. **78**, 033604 (2009).
[5] e.g. Finley et al., PRL **98**, 265301 (2007).

Actually this is too pessimistic. What saves us is that in a thin superconducting film ($d \ll \lambda_{\rm L}$) the Meissner effect itself is profoundly modified; in fact, for large enough r the supercurrent, and hence the vortex-antivortex interaction, falls off only as r^{-2}, not exponentially. This by itself would not be enough to restore KT-type behavior. What is more important is that the "effective London penetration depth" up to which the supercurrent falls off as r^{-1} as in the neutral case, is itself very greatly increased with respect to its bulk value $\lambda_{\rm L}$:

$$\lambda_{\rm eff}^{(2{\rm D})} = 2\lambda_{\rm L}^2/d \tag{14}$$

For a film of thickness $\sim$ a few tens of Å, this length is very long; it can even be of the order of sample dimension or larger. Thus, for a sufficiently thin and dirty[6] film the analogy to the case of a neutral superfluid is essentially exact, the only difference being that the quantum of circulation is now $\hbar/2m_{\rm e}$ rather $\hbar/m_{^4{\rm He}}$. Thus one would expect that the KT transition temperature should be given by the formula

$$k_{\rm B}T_{\rm KT} = \frac{\pi}{8}\rho_s^0(T)(\hbar/m)^2 \quad \text{(not } \pi/2) \tag{15}$$

where m is the electron mass. It should be noted that for a dirty superconductor $\rho_s^0(T)$, the "mean-field" superfluid density, may already be much smaller (by a factor $\sim l/\xi_0$, l = normal-state mean free path, ξ_0 = Cooper-pair radius) than $\mathcal{O}(\rho(1-T/T_c))$. The condition for the KT transition to be well separated from the mean-field (BCS) transition is therefore approximately $d \lesssim \lambda_T(\xi_0/l) \gg \lambda_T$ ($\lambda_T \equiv$ de Broglie wavelength at T_c), thus much weaker than for ^{4}He.

The application of KT-type ideas to a thin superconducting film has been discussed by Halperin and Nelson,[7] with the following conclusions:

(1) Above T_c, just as in the neutral superfluid, the "superconducting" contribution to the *dc* conductivity[8] comes entirely from the motion of free vortices. A simple calculation gives

$$R = n_f \mu e^2/\hbar\pi^2 \tag{16}$$

where μ is the vortex mobility and n_f the number of *free* vortices per unit area. For a dirty superconductor the theoretical expression for μ is $2e^2\xi_{\rm GL}^2/\hbar^2\pi\sigma_{\rm n}$ where $\sigma_{\rm n}$ is the normal-state resistivity. Using this and the fact that $n_f \sim \xi_+^{-2}$, we find for the conductivity relative to the normal-state (i.e. $T > T_{\rm BCS}$) value the result

$$\sigma/\sigma_{\rm n} \sim \xi_+^2/\xi_{GL}^2 \tag{17}$$

which diverges at the KT transition (as $\exp t^{-1/2}$) since $\xi_{\rm GL}$ is finite there.

(2) HN also show that the diamagnetic susceptibility diverges as $T \to T_{\rm KT}$ from above;

$$\chi \sim \text{const.}\,\xi_+^2(T) \sim \exp t^{-1/2} \tag{18}$$

[6] Dirt helps by increasing the value of $\lambda_{\rm L}$.

[7] JLTP **36**, 599 (1979).

[8] Or rather to the dc resistivity, which would be zero in the absence of vortex motion!

(3) Finally, in superconductors there is a regime analogous to what we called regime (a) for the neutral superfluid, that is, the regime of low (in fact zero) frequency and high currents. In that case, by analogy to the theory developed by AHNS for the neutral superfluid, which we recall gave a vortex-antivortex unbinding rate proportional to v_s^4, we find that the number of unbound vortices $\propto v_s^2$. Since the voltage should be proportional to the number of unbound vortices times the drift rate across the current, which is proportional to v_s, we find below T_c–up to logarithmic factors the result

$$V \propto \text{const. } I^\delta \tag{19}$$

where the exponent δ is predicted to approach 3 in the limit $T \to T_{\text{KT}}^+$.

A test of these ideas was carried out by Hebard and Fiory[9] (and a number of other people); they measured the dc conductivity of a 100 Å thick film of In/InO as a function of temperature, in a magnetic field less than 10^{-6} T (necessary so that the essential condition of "charge neutrality" (equal number of vortices and antivortices) should be realized). They verified essentially all the predictions of the theory which they were able to test, in particular the $\exp -bt^{-1/2}$ behavior of the resistance above T_{KT} (see fig. 3 of their paper) and the power-law behavior of the $I-V$ curve at T_{KT} (though the measured value of δ is ~ 3.4 rather than 3: part but not all of the discrepancy is thought to be understood, see p. 1606, paragraph 2).

The good agreement of the dc conductivity data for dirty thin superconducting films with the KT theory is actually somewhat surprising, since that theory relies, for not only dynamic but also static predictions, on the assumption that the vortices are free to move in the plane, and one might have thought that in a dirty film they would be pinned by disorder. It seems that either the large size of the core ($\sim \xi_0 \sim 50$ Å in In/InO$_x$) or some other consideration makes them insensitive to pinning on the scale of atomic distances.

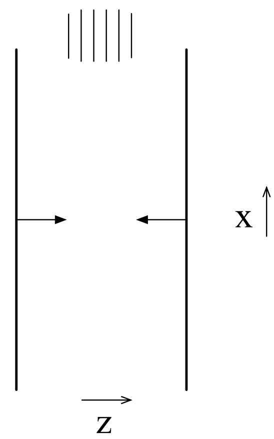
A system somewhat related to superconducting films is a planar array of superconducting islands coupled by Josephson junctions, and a number of experiments have looked for the KT transition in this system.[10] One advantage it possesses vis-à-vis metallic films is that the energy scale of the mean-field behavior is set by the 3D bulk energy gap Δ, while the "superfluid density" is controlled by the inter-island Josephson coupling J, and there is no reason for there to be any particular relation between Δ and J. Since the effective superfluid density $\rho_s(T)$ is proportional to J, and T_{KT} is in turn controlled by $\rho_s(T)$, while the mean-field transition temperature T_{c0} is controlled by Δ, this means that it is very easy to obtain the condition $T_{\text{KT}} \ll T_{c0}$, which inter alia allows us to treat the normal fraction as negligible. Generally speaking, the agreement with the BKT predictions for the (linear and nonlinear) $I-V$ characteristics of those systems is comparable to, though perhaps somewhat less impressive than, that on thin films. In addition, the application of a magnetic field perpendicular to the array introduces a

[9] PRL, **50**, 1603 (1983).

[10] See e.g. C.J. Lobb, Physica **126**B, 319 (1984).

whole new gamut of effects, since the Josephson coupling is now "frustrated": see Lobb, loc. cit.

Finally let us discuss briefly some more recent work on the KT transition in quasi-2D ultracold atomic gases.[11] Actually, when viewed in a realistic experimental context these systems immediately raise a conceptual problem, for the following reason: Everything we have said so far relies implicitly on the idea that the "2D" system in question does not possess true long-range order, but at the best an order (below T_{KT}) of "topological" nature. That this assumption is indeed true for a truly 2D system in free space is guaranteed by Hohenberg's theorem. However, in real life the ultracold atomic gases are always confined by a harmonic trapping potential, typically provided by an inhomogeneous magnetic field; in particular, this is the case in the experiments of Hadzibabic et al. But, while the concept of long-range order may itself be somewhat ill-defined in the presence of a spatially inhomogeneous potential, one can still ask whether "BEC" occurs, i.e. whether the single-particle density develops a single eigenvalue of the order of the total particle number N; and for an *ideal* 2D gas in a (2D) harmonic potential, the answer is certainly yes! (The reason is that the single-particle density of states is thinned down from its constant form for 2D free space to a form linear in ϵ, which is "thin" enough to guarantee the appearance of BEC). Thus, at first sight, KT-type arguments cannot get off the ground. On the other hand, it is tempting to argue that with a repulsive interatomic interaction Hartree-type effects will make the chemical potential $\mu(\mathbf{r})$ seen by the "last added" atoms look fairly flat over the region of cloud, so that the behavior of fluctuations around the mean-field will be much as in 2D free space. At least an experiment seems required...



One feature of the ultracold atomic gases which distinguishes them qualitatively from the more traditional condensed-matter systems discussed so far is the possibility of obtaining more or less direct information, by interference experiments, about the *phase* of the condensate wave function. In a famous early experiment along these lines two initially isolated bulk (3D) condensates were allowed to expand and overlap, and a spectacular interference pattern was observed, with however an offset which varied randomly from shot to shot, indicating that in some sense the measurement forced the two condensates to "choose" a definite relative phase. Suppose now that we have two quasi-2D condensates occupying nearby parallel xy-planes, and they are initially isolated but subsequently released and allowed to expand and overlap along the z-axis. Naively, we expect that the condensate phase difference $\Delta\phi(\mathbf{r}_{\|})$ for *any one* value of $\mathbf{r}_{\|} \equiv (x, y)$ would be chosen randomly. However, once that is fixed the value of $\Delta\phi(\mathbf{r}'_{\|})$ for any other point $\mathbf{r}' \equiv (x', y')$ would be (partially) fixed by the phase correlations within the individual planes. Since the position of the fringes along the z-axis is fixed by $\Delta\phi(\mathbf{r}_{\|})$, it follows that a measurement of the density $\rho(z, \mathbf{r}_{\|})$ integrated (partially or totally, see below) over $\mathbf{r}_{\|}$ will give us information about the phase correlations within the individual planes. More quanti-

[11] Hadzibabic et al., Nature, **441**, 1118 (2006).

tatively, on any one shot, the density $\rho(z, \mathbf{r}_\parallel)$ should be given (with a suitable choice of origin for z) by

$$\rho(z, \mathbf{r}_\parallel) = \text{const.}\ \big(A + B\cos(k_0 z + \Delta\phi(\mathbf{r}_\parallel))\big) \tag{20}$$

where the quantity $B/(A+B) \equiv V$ is the "local" fringe visibility, and $k_0 = 2\pi/\lambda_f$ where λ_f is the fringe spacing (determined by the detailed dynamics of the expansion process). What is actually measured in the experiment of Hadzibabic et al. (by laser absorption imaging) is the y-integral of the density (20), and they represent the result in the form (apart from an uninteresting envelope function)

$$C(x, z) = 1 + c(x)\cos(k_0 z + \phi(x)) \tag{21}$$

They then consider the quantity[12] (averaged over tests)

$$\langle C^2\rangle(L_x) = L_x^{-2}\left|\int_0^{L_x} dx\, c(x)\exp i\phi(x)\right|^2 \tag{22}$$

A little thought shows that provided V and k_0 are independent of x and y, the quantity $\langle C^2\rangle(L_x)$ is proportional, at any given temperature, to the double integral

$$\begin{aligned} L_x^{-2}\int_0^{L_x} dx \int_0^{L_y} dy\ \left|\langle\exp\big(i\Delta\phi(x,y) - i\Delta\phi(x',y')\big)\rangle\right|^2 \sim \\ \frac{1}{L_x}\int_0^{L_x} dx \int_0^{L_y} dy\ |\langle\psi(0,0)\psi^*(x,y)\rangle|^2 \end{aligned} \tag{23}$$

the dependence on the "average" phase difference of the two planes having fallen out. Note that in this expression L_y is constant at the physical y-dimension of the cloud (about $10\,\mu$). Most of the data was taken for $L_y \ll L_x \equiv x$-dimension of cloud ($\sim 159\,\mu$).

Let us consider the dependence of $\langle C^2\rangle(L_x)$ and L_x under these conditions for three cases, denoting the correlation $\langle\psi(0,0)\psi^*(x,y)\rangle$ (assumed isotropic) by $K(r_\parallel)$ ($r_\perp \equiv (x^2+y^2)^{1/2}$):

(1) True LRO ($K(r) \to$ const., $r \to \infty$): evidently $\langle C^2\rangle(L_x) =$ independent of L_x.

(2) Short-range correlations only ($K(r) \sim \exp -r/\xi$ where $\xi \ll L_x$): $\langle C^2\rangle(L_x) \sim L_x^{-1}$.

(3) Power-law correlations, $K(r) \sim r^{-\eta}$: $\langle C^2\rangle(L_x) \sim L_x^{-2\eta}$

Hadzibabic at al. find that above a certain temperature T_0 $\langle C^2\rangle(L_x)$ is approximately proportional to L_x^{-1} (in their notation $\alpha \approx 0.5$) while below T_0 it scales as $L_x^{-2\alpha}$ where α tends to 0.25 as $T \to T_0$ from below. This is exactly what is expected at the KT transition, since $\eta \to 1/4$ as $T \to T_{\text{KT}}^{(-)}$. They thus tentatively identify T_0 with T_{KT}.

A further, more qualitative, observation reported by Hadzibabic et al. is of free vortices which manifest themselves as "dislocations" in the interference pattern (note that tightly bound vortex-antivortex pairs would not show up in the pattern). As indicated in their fig. 4, such free vortices are virtually absent for $T < T_0$ but proliferate rapidly above T_0, again in qualitative agreement with the BKT scenario.

[12] They do not actually define $\langle C^2\rangle(L_x)$ explicitly: this is my best guess as to the implied definition. (cf. their eqn. (1))