## Lecture 4   Single-particle QM in 1, 2 and 3D

Consider a single nonrelativistic particle of mass $m$ moving in a $d$-dimensional conservative, velocity-ind$^t$ pot$^l$ $V(\underset{\sim}{r})$ in a stationary state it satisfies the TISE*

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\underset{\sim}{r})\right)\psi(\underset{\sim}{r}) = E\psi(\underset{\sim}{r}) \qquad \nabla^2 \equiv \sum_{i=1}^{d} \partial^2/\partial x_i^2 \qquad (*)$$

The same eq$^n$ applies to two particles of mass $m_1, m_2$ moving in a mutual pot$^l$ $V(\underset{\sim}{r}_1 - \underset{\sim}{r}_2)$, provided that $\underset{\sim}{r}$ is interpreted as the relative coord. $\underset{\sim}{r}_1 - \underset{\sim}{r}_2$ (the COM wf factors out) and $m$ is replaced by the reduced mass $\mu \equiv (m_1^{-1} + m_2^{-1})^{-1}$. We will consider for simplicity (in 2D and 3D) only the case of a central pot$^l$, $V(\underset{\sim}{r}) = V(|\underset{\sim}{r}|)$. (In 1D, analogous condition is $V(x) = V(-x)$)

### 3D problem (recap)

### A. The bound-state problem

We must solve the TISE (*) subject to the conditions (a) $\psi(\underset{\sim}{r}) \to 0$ for $|\underset{\sim}{r}| \to \infty$ in any direction, (b) $\psi(\underset{\sim}{r})$ cannot behave as negative power of $r$ for $\underset{\sim}{r} \to 0$ unless the potential is infinite at the origin (eg $\delta$-f'n or Coulomb). The first condition evidently $\Rightarrow E < 0$.

(b. which crudely speaking imply)

Standard procedure: separation of variables

$$\psi(\underset{\sim}{r}) = R_l(r)\,Y_{lm}(\theta, \varphi)$$

then the radial wf obeys the diff$^l$ eq$^n$

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{dR_l}{dr}\right) + \left(V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right)R_l(r) = E R_l(r) \qquad ($$

If we introduce $\chi_l(r) \equiv r R_l(r)$, then $\chi_l$ satisfies the 1D-like eq$^n$.

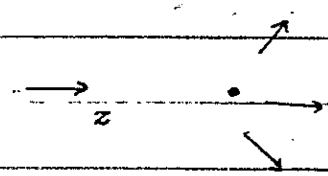
* with b.c.'s: (a) single-valued (b) square-f'ble (c) gradient continuous except at points where $V$ infinite.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\chi_l(r) + \left\{V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right\}\chi_l(r) = E\chi_l(r)$$

The boundary condition at $\infty$ is simple ($\chi_l(r) \to 0$); however, the b.c. at the origin needs some care. If for example $\chi_l(r) \to$ const. as $r \to 0$, then $R_l(r) \sim 1/r$ and deriv. is discontinuous at $r = 0$. This is allowed only for $V(0) = \infty$ (e.g. ie $\delta$-function) ~~or [illegible]~~ Thus, if $V$ is finite at the origin $\chi_l(r)$ must tend to zero at least as $r$. This is actually ensured automatically for the case $l \neq 0$ (since the asymptotic form is $r^l$ or $r^{-(l+1)}$ and the second must be excluded), but for the s-wave case it must be imposed explicitly. Thus the case of s-wave scattering in a 3D central potential is not equivalent to a 1D problem with $V(r) \to V(x)$. In particular, there is no general theorem that for a pot[l] which is everywhere attractive a bound state must exist; in fact, in specific cases (e.g. 3D square well) it is straightforward to show that the well must be of a certain minimum depth/extent to bind a particle.

B. Scattering (e.g. LL* §122)

Statement of problem:




incoming wave $\sim \exp ikz$

We are interested in prob. of observing scattered particle at $\infty$ in dir[n] $(\theta, \varphi)$.
Apart from const. factor of $k$, and geom[l] factor $1/r^2$, this is prop. to $|\psi(r)|^2$ at $(r\theta\varphi)$ for $r \to \infty$.

Standard procedure: split w.f. into partial-wave comps:

$$\psi(\mathbf{r}) = \sum_{lm} c_{lm} Y_{lm}(\theta\varphi) R_{lm}(r)$$

then each partial wave satisfies (†) but now with positive $E$ ($\equiv \hbar^2k^2/2m$). If we choose z-axis along dir[n] of incidence, then since pot[l] is central all terms with $m \neq 0$ vanish and we are left with

$$\psi(r) = \sum_l c_l P_l(\theta) R_l(r) \qquad (‡)$$

Split $\psi$ into "incoming" and "outgoing" waves:

* 1965 ed[n]

The functions $R_l(r)$ must satisfy the eqn (†), which for sufficiently large $r$ (such that $V(r) \to 0$ and $k^2 r^2 \gg l(l+1)$) becomes simply

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr} r^2 \frac{dR_l}{dr} + k^2 R_l(r) = 0$$

or in terms of $\chi_l(r) \equiv r R_l(r)$,

$$\frac{d^2\chi_l(r)}{dr^2} + k^2 \chi_l(r) = 0$$

The general solution is obviously an arbitrary linear combination of an outgoing wave $e^{ikr}$ and an incoming one $e^{-ikr}$. It is convenient to write this comb.$^n$ in the form

$$\chi_l(r) = \frac{(2l+1)}{k} A_l \sin(kr - l\pi/2 + \delta_l) \qquad (A_l \text{ complex}) \qquad (3$$

(note $\delta_l \equiv \delta_l(k)$ !)

where the overall magn takes care of the constant $c_l$ in (‡). Then the complete form of $\psi(\underset{\sim}{r})$ in the limit $r \to \infty$ is

$$\psi(\underset{\sim}{r}) = \sum_{l=0}^{\infty} (2l+1) A_l P_l(\cos\theta) \frac{\sin(kr - \frac{1}{2}l\pi + \delta_l)}{kr}$$

However only part of this corresponds to the scattered wave. If we write

$$\psi(r) = e^{ikz} + \psi_{sc}(r)$$

then $\psi_{sc}$ must contain only an "outgoing" part (and this must be true for each p.w. compt. separately). Now we have for $r \to \infty$

$$e^{ikz} = e^{ikr\cos\theta} \to \sum_l (2l+1) i^l P_l(\cos\theta) \sin(kr - \tfrac{1}{2}l\pi)$$

so this condition gives

$$A_l = i^l e^{i\delta_l} \qquad (\cancel{\phantom{xx}}$$

and the coefficient of the $l$-th outgoing partial wave is

$$f_l \equiv e^{2i\delta_l} - 1$$

Hence, finally, the complete outgoing wave has the form for $r \to \infty$

$$\psi_{sc}(\underset{\sim}{r}) = f(\theta) \frac{e^{ikr}}{r}, \qquad f(\theta) = \sum_{l=0}^{\infty} (2l+1) \frac{(e^{2i\delta_l} - 1)}{2ik} P_l(\theta)$$

and the scattered intensity is $|f(\theta)|^2/r^2$.

The total scattering cross-section $\sigma$ is $\int |f(\theta)|^2 d\Omega$, thus ($\int P_l(\theta)P_{l'}(\theta)\,d\Omega = \frac{4\pi}{2l+1}\delta_{ll'}$)

$$\sigma = \frac{1}{4}\sum_{l=0}^{\infty}(2l+1)\frac{|e^{2i\delta_l}-1|^2}{k^2} = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l(k)$$

Hence the phase shifts $\delta_l$ ($\equiv \delta_l(k)$) give complete infⁿ on the scattering. They are obtained by solving the complete equation, valid for all $r$,

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\chi_l(r) + \left\{V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right\}\chi_l = E\chi_l(r)$$

and subject to the b.c. at $r=0$, and matching to the asymptotic form (3) in the limit $r\to\infty$.

A particularly interesting case arises when $kr_0 \ll 1$ where $r_0$ is the range of the potˡ $V(r)$. In this case, since $\chi_l \sim R_l \sim (kr)^l$ for $kr \ll 1$, the partial waves with $l \neq 0$ never "feel" the potential, and for them $\delta_l \cong 0$. Thus the only relevant p.w. is the s-wave compt. In this case it is easier to proceed as follows: ~~we simply solve the~~ Assume for the moment that the scattering is not too "strong" (see below), and consider the limit ~~$kr_0 \to 0$, or $k \to 0$~~ but still $k \to 0$. Then in the region $kr \ll 1$ but $r \gg r_0$, the TISE reduces simply to the Laplace eqn $\nabla^2\psi = 0$, and the most general form of the s-wave (spherically symmetric) solution is, apart from normalizⁿ:

$$\psi = \psi(r) = 1 - a_s/r$$

it can be +ve or −ve

where $a_s$ is def as the zero-energy s-wave sc. length. Under most circumstances, e.g. the value of $|a_s|$ is comparable to the ~~max~~ range $r_0$ of the potential; however, close to the onset of a bound state it can be $\gg r_0$. In fact, as we approach the BS from above, $a_s \to -\infty$, and as we approach it from below (i.e. as the BS sinks into the continuum) $a_s \to +\infty$.

From the point of view of the scattering problem, a given value of $a_s$ is equivalent to the effect of a $\delta$-function pseudopotential

$$\boxed{V_{ps}(r) = \frac{2\pi\hbar^2 a_s}{m}\delta(\underline{r})} = g\,\delta(\underline{r})$$

The simplest way to see this is to note that the general form of the s-wave solution of ~~the~~ the ~~TISE~~ zero-energy TISE

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{2\pi\hbar^2}{m}a_s\,\delta(r)\,\psi(r) = 0$$

is precisely const $- a_s/r$, and ~~therefore~~ we must fix the constant to give the right result for $r \gg a_s$, so in general the above form $\psi(r) = 1 - a_s/r$. It is also straightforward to show that the (low-energy) energy shift of a particle confined in a spherical box with a pot'l near the origin s.t. the zero-E sc. length is $a_s$ is just $(2\pi\hbar^2 a_s/m)\,|\psi(0)|^2$. (where $\psi(0)$ is calculated in the absence of the pot'l). Alternatively we can say that in 3D, for a $\delta$-function potential of strength $g$, the sc. length ~~is~~ $a_s \propto g$.

Finally, we note that from the above definitions $a_s \equiv f_0$ and hence, by eqn. (†

$$\boxed{\delta_0(k) = k a_s} \quad (k \to 0)$$

## 1 dimension

### A. Bound states

In 1D the TISE is simply

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x)$$

We will not necessarily assume that $V$ is parity-inv't. (i.e. $V(x)=V(-x)$). The b.c.'s are that $\psi(x)$ is finite, single-valued, square-∫'ble and has a continuous first derivative except poss. at points where $V(x)$ is infinite. The square-∫'bility condit clearly implies that $\psi(x) \to 0$ as $|x| \to \infty$, which in turn implies $E<0$.

We will show that for any form of $V(x)$ s.t. $V(x) \le 0$ everywhere, at one bound state always exists.

Proof: Consider some range $R$ of $x$, say $-a < x < a$, ~~where $V(x)$ is~~ s.t. it is not true that $V(x) = 0$ everywhere in $R$. Construct the variational w.f.

$$\psi(x) = \overset{A}{\cancel{\text{const}}}, \quad x \in R$$

$$\psi(x) = A \exp -|x|/\lambda^{2}, \quad x \notin R.$$

The normaliz$^n$ condition then $\Rightarrow$ $A = [2(a+\lambda)]^{-1/2}$. The (exp. value of the) energy of the above state is composed of pot$^l$ and kinetic terms: since the pot$^l$ contr$^n$ from $x \notin R$ is ~~$\langle E \rangle$~~ negative (or zero) we have the upper limit

$$\langle E \rangle \le A^2(-V_0 + \tfrac{1}{4}\lambda^{-1}) = \cancel{\tfrac{1}{2}}(-|V_0| + \tfrac{1}{4}\lambda^{-1})/(a+\lambda) \qquad (*)$$

where $\lambda^{-1} \left( \times \frac{\hbar^2}{2m} \right)$

$$|V_0| \equiv \cancel{\tfrac{1}{2}} \left| \int_{-a}^{a} V(x)\,dx \right|$$

It is clear that however small $|V_0|$, we can always find a value of $\lambda$ ~~small~~ large enough that the RHS of $(*)$ is $< 0$. Thus at least one b.s. always exists, QED.

$*$ To see why an analogous result doesn't follow in 3D, imagine imposing the condition that $\psi(x)$ vanish at some point, say $x = 0$. Then the minimum kinetic energy required within the range $-a < x < a$ is of order $A^2 \cdot \hbar^2/2ma$ ~~$a^{-1}$~~; effectively this adds a constant $a^{-1}$ to the $\lambda^{-1}$, ~~[illegible]~~ in the numerator of $(*)$, and it can no longer nece$^y$ be made $-$ve.

[Prob. on odd-parity state]

## Scattering

We consider scattering of an incident wave $e^{ikx}$ from $-\infty$, by a potential $V(x)$ (not necessarily symmetric) which occupies a finite region of the x-axis around $x = 0$.

Since in the potential-free region the solution of the TISE must satisfy $d^2\psi/dx^2 + k^2\psi = 0$ ($k \equiv (2mE/\hbar^2)^{1/2}$), the most general solution is (since there is no wave incident from $x = +\infty$)

$$\psi(x) = e^{ikx} + re^{-ikx}, \quad x < 0$$

$$= te^{ikx} \qquad x > 0$$

where $r$ and $t$ are the (complex) reflection + transmission coefficients. From the conservation of current it is clear that

$$|r|^2 + |t|^2 = 1$$

We can also consider the case of a wave incident from $+\infty$ in which case the reflection and transmission coefficients are defined to be $r'$ and $t'$ respectively. In fact, we can write down a "scattering matrix"

$$S = \begin{array}{c c} & \begin{array}{cc} + & - \end{array} \\ \begin{array}{c} + \\ - \end{array} & \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} \end{array}$$

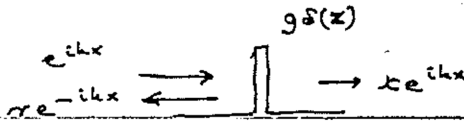
Since this matrix must be unitary (so as to preserve the normalization of an arbitrary comb$^n$ of $e^{ikx}$ and $e^{-ikx}$), we have $SS^\dagger = 1$, ie

$$|t|^2 + |r|^2 = |t'|^2 + |r'|^2 = 1 \qquad \text{(as before)}$$

$$t/r = -(t'/r')^* \quad \Rightarrow \quad |t| = |t'|, \ |r| = |r'|$$

Note that the phase of $t$ is physically meaningful (eg we could interfere the transmitted beam with one which has avoided the scatterer) but the phase of $r$ is essentially a question of the choice of origin, so without loss of generality we can take $r$ real. For a symmetric potential ($V(x) = V(-x)$) we can take $r = -r'^*$ and $t = t'^*$, so the problem is completely specified by the single complex no. $t$. For the more general case, however, $t$ may differ from $t'$ in phase (though not in amplitude) [Prob.] Evidently we can define a "scattering length" $a$ by $t = |t|e^{-ika}$.

It is interesting to study the special case of a $\delta$-function potential, $V(x) = g\delta(x)$. Recall that in 3D, for a potential $g\delta(\mathbf{r})$, we had a low-energy s-wave scattering length $a_s = g/(2\pi\hbar^2/m)$. Now in 1D ~~we can define a scattering length~~ (quick) the dimensions of $g$ are different ($EL$ as opposed to $EL^3$) so it is obvious on these grounds alone that the above relation does not hold. So what is the relation between $g$ and $a$?




We match the wave function itself at $x = 0$, giving

$$r = t - 1$$

Integration of the TISE across the origin gives

$$\Delta\left(\frac{\partial\psi}{\partial x}\right) \left(\equiv \frac{\partial\psi}{\partial x}(x=0+) - \frac{\partial\psi}{\partial x}(x=0-)\right) = +\frac{2mg}{\hbar^2}\psi(0)$$

~~i.e.~~ but

$$\Delta(\partial\psi/\partial x) = ik(t-(1-r)) = 2ik(t-1) \qquad , \psi(0) = t$$

~~so substituting the above~~

so:

$$ik(t-1) = (mg/\hbar^2)\,t$$

$$\Rightarrow \qquad \boxed{t = \frac{1}{1 - mg/ik\hbar^2}} \qquad (\text{note } |t|<1\,!)$$

and $\to 0$ as $k \to 0$.

If we define "even" and "odd" scattering amplitudes by

~~$f_{even}$ $f_{odd}$ $\equiv$ ($\psi$ ... ($\psi(x>0) \pm \psi(x<0) - e^{ikx}$)~~

~~then from $1+r = t$, $f_{odd} = 0$ and $f_{even} = t-1$~~

$$\psi_{sc} \equiv \psi - e^{ikx} \equiv f_{even}(e^{ikx} + e^{-ikx}) + f_{odd}(e^{ikx} - e^{-ikx})$$

then from $1+r=t$ we have $f_{odd} = 0$,

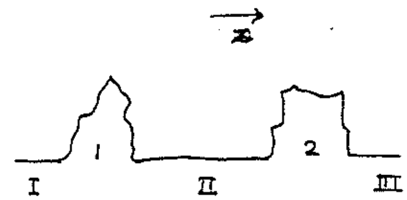
$$f_{even} = t - 1 = \frac{1}{1 - \frac{ik\hbar^2}{mg}} \equiv \frac{1}{1 + ika_{1D}}$$

where the "1D sc. length" $a_{1D}$ is def by

$$\boxed{a_{1D} \equiv -\frac{\hbar^2}{mg}}$$

so $a_{1D}$ is inversely prop to $g$ and opposite in sign.

## Two barriers in series




It is worth discussing this problem here. It will be needed for the discussion of localiz$^n$ in L.6. Suppose that there are two distinct barriers separated by a region of zero potential (II); note that the length of II does not have to be large c'd the width of the barriers, all we need is to be able to define an "asymptotic" behaviour.

Suppose a particle is incident from $z = -\infty$ with momentum $\hbar k$, i.e. the incoming wave is $\exp ikz$. Then in the various $V = 0$ regions we have

$$\mathrm{I}: \quad \psi(x) = \exp ikz + A \exp -ikz$$

$$\mathrm{II}: \quad \psi(x) = B \exp ikz + C \exp -ikz$$

$$\mathrm{III}: \quad \psi(x) = D \exp ikz$$

so that the transmission coefficient is $|D|^2$ and the refl$^n$ coeff. $|A|^2$. How are these quantities related to the reflection and transmission ~~amplitudes~~ coefficients of the two barriers?

To see this, we note that

$$\left.\begin{aligned} A &= r_1 + t_1' C \\ B &= t_1 + r_1' C \\ C &= r_2 B \\ D &= t_2 B \end{aligned}\right\} \quad \Rightarrow \quad B = \frac{t_1}{1 - r_1' r_2} \qquad \Rightarrow \quad D = \frac{t_1 t_2}{1 - r_1' r_2}$$

The total transmission coefficient $T_{12}$ ~~(through)~~ $\equiv |D|^2$ is therefore given (since $|t_1|^2 = T_1$, $|r_1'|^2 = R_1$ etc.) by the expression

$$T_{12} = \frac{T_1 T_2}{|1 - r_1' r_2|^2}$$

In evaluating the quantity $r_1' r_2$, we must bear in mind that it includes phase shift $2\varphi$, where $\varphi = ka$, $a$ being the difference between the "origins" chosen for barriers 1 and 2. (Recall that the phase of $r$ is a matter of convention which depends on this origin). Thus, defining $\theta \equiv 2\varphi + \arg($

we find $(1 - r_1' r_2) = 1 - \sqrt{R_1 R_2}\, e^{i\theta}$ and hence

$$\boxed{T_{12} = \frac{T_1 T_2}{1 + R_1 R_2 - 2\sqrt{R_1 R_2}\cos\theta}}$$

which is the required result. Using the relations $t_1/r_1 = -(t_1'/r_1')^*$ etc. (2.4), it may be verified with some labor that $R_{12} = 1 - T_{12}$, as of course it must be.

Note that in the limit of large reflectances, $T_{12}$ can actually be _larger_ than $T_1 T_2$ (cf the situation in a Fabry-Perot etalon).

## 2 dimensions

Note that the problem of scattering (or BS) in 2D is relevant not only to "truly" 2D systems, but to 3D ones with cylindrical symmetry (e.g. a charged particle in the field of a long straight charged wire). The 2D problem is considerably trickier than the 1 or 3D ones ~~since~~, in part bec. there is no limit in which it can be reduced to an essentially 1D one.

The Schrödinger eqn. in polar coords $r, \varphi$ can be separated by writing

$$\psi(r) = \exp i n\varphi \, R_n(r) \qquad n = 0, \pm 1, \pm 2 \ldots \quad \text{(but note}$$

The radial wave function now obeys

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d}{dr}\, r\frac{dR_n}{dr}(r) + \frac{n^2\hbar^2}{2mr^2}R_n(r) = ER_n(r) \quad (-VR_n(r))$$

Unlike the case of 3D, there is now no transfn which reduces this to a simple 1D TISE. The most obvious substitution is $\chi_n(r) = r^{1/2}R_n(r)$, but this gives

$$-\frac{\hbar^2}{2m}\frac{d^2\chi_n}{dr^2} + (n^2 - 1/4)\frac{\hbar^2}{2mr^2}\chi_n = E\chi_n \quad (- V(r)\chi_n)$$

and no simple transformation will get rid of the 1/4.

However, it is amusing that we can cancel this term by application of an appropriate AB flux. Suppose the system is pierced by a thin tube containing a flux $\Phi$, s.t. no field leaks out into the region where the particle is. We describe this by a magnetic vector potential $\underset{\sim}{A}(r) = \hat{\underset{\sim}{\varphi}} \times \Phi/2\pi r$, and the effect is to replace $-i\partial/\partial\varphi$, in the angular part of the KE, by $-i\,\partial/\partial\varphi - (e/\hbar)A_\varphi = i\,\partial/\partial\varphi - (e/h)\Phi$. As before, to ensure single-valuedness the angular wave function must be of the form $e^{in\varphi}$, but now the associated KE is not $n^2\hbar^2/2mr^2$ but rather $(\hbar^2/2mr^2)(n - \Phi/\Phi_0)^2$, where $\Phi_0$ is the (s-p) flux quantum $h/e$. We see that if $\Phi$ is exactly half a flux quantum and $n = 0$, then the flux term exactly cancels the 1/4 in (*) and we indeed get the simple "1D" form of the radial wave equation, just as in the 3D case

The theory of scattering in 2D can be developed similarly to the 3D case*. In this case the free-space solutions of the radial SE (*) are the Bessel and Neumann functions† $J_n(kr)$, $N_n(kr)$, and the appropriate circular expansion of an incoming wave is

$$\exp ikz = \sum_{n=-\infty}^{\infty} i^n \exp in\varphi \, J_n(kr)$$

There is no great point in going through the details, but note that the s-wave phase shift $\delta_0(k)$ for $k \to 0$, (which, ~~and~~ in 3D is $ka$, and in 1D $-ka_{1D}$, cf. above) in 2D diverges logarithmically:

$$\delta_0^{(2D)}(k) \cong \frac{1}{2\pi} \ln (k_s \tilde{a})$$

where $\tilde{a}$ is a characteristic length of the potential, which however should probably not be thought of as a scattering length. (~~The~~

Bound states: We have seen that a purely attractive potential (or more generally one whose space integral is $< 0$) will always ~~[illegible]~~ sustain at least one BS in 1D, but not necessarily in 3D. What is the result in the 2D case? Because of the rather awkward properties of the Bessel and Neumann functions, it is actually easier to look at the problem in $\underline{k}$-space rather than $\underline{r}$-space. ~~The~~ The k-space form of the TISE is (in any dimension)

$$(\epsilon_k - E)\psi_k = -\sum_{k'} V_{kk'} \psi_{k'} \qquad (V_{kk'} \equiv V_{k-k'})$$

Let's consider a very extended state, such that all k's are $\ll r_0^{-1}$, the inverse range of the potential. Then ~~we~~ we should be able to replace $V_{kk'}$ by the constant $V_0 \equiv \int V(r)\,dr$, ~~so~~ and the TISE reduces to

$$1 = -V_0 \sum_k (\epsilon_k - E)^{-1}$$

* S. Adhikari, Am. J. Phys. 54, 362 (1986): M. Randeria et al., PR B 41, 327 (1990)

~~It is clear that, whatever it, this expr. has~~ .

† Abramowitz & Stegun ch. 9. The function $N_n(x)$ is often denoted $Y_n(x)$ and called a Bessel function of the second kind.

It is clear that, whatever $d$, this eqn has no bound-state ($E<0$) solution if $V_0 > 0$. If $V_0 < 0$, then everything depends on the DOS as $k$ (or $\epsilon \to 0$). The sum over $k$ becomes in $d$ dimensions $C_d \int d\epsilon\, \epsilon^{(d-1)/2}$, where $C_d$ is a constant; in particular, $C_2 = (m/2\pi\hbar^2)$. Thus the TISE reads

$$1 = |V_0|\, C_d \int^{\epsilon_c} d\epsilon \frac{\epsilon^{(d-1)/2}}{\epsilon - E} \qquad (V_0 < 0)$$

It is clear that this eqn always has an $E<0$ solution for $d = 1$; for $d = 3$ it may or may not depending on the high-energy cut off $\epsilon_c$ (which must be determined e.g. by the departure of $V_{k-k'}$ from the constant value $V_0$). For $d = 2$ a solution always exists, but the binding energy is exponential small for $V_0 \to 0$:

$$E \approx \epsilon_c \exp - 1/(mV_0/2\pi\hbar^2)$$

(Cf. the solution of the Cooper problem in superconductivity theory, which is formally identical to a simple 2D Schrödinger problem since the relevant DOS is similarly a constant)