The $\nu = 5/2$ fractional quantum Hall effect [1]

As we have seen, the vast majority of quantum Hall plateaux observed experimentally, whether in Si MOSFET's, GaAlAs-GaAs heterostructures or graphene, occur either at integral values of the filling factor $\nu \equiv N_e/N_\phi$ ($N_e \equiv$ number of electrons, $N_\phi \equiv$ number of flux quanta) (IQHE) or at rational fractions $\nu = p/q$ with q an *odd* integer (FQHE). Within the composite-fermion picture, the FQHE is explained by setting $p = k$, $q = 2nk \pm 1$ and viewing the FQHE as derived from the IQHE with $\nu = k$ by attaching to each electron $2n$ imaginary "flux quanta"; or, what is equivalent, multiplying the IQHE wave function by $(z_i - z_j)^{2n}$. However, as we have seen in lecture 20, in the case of $\nu = 1/2$ the same procedure yields the conclusion that the "parent" IQHE state should correspond to $\nu = \infty$, i.e. it should be a Fermi liquid. Moreover, we saw that experiments, in particular on magnetic focusing, seem consistent with this point of view.

Prima facie, one would expect the states which occur at $\nu = n + 1/2$ in traditional QHE systems such as GaAlAs-GaAs heterostructures[2] to differ from that at $\nu = 1/2$ only by having n Landau levels completely filled; the situation in the $(n + 1)$-th LL should be similar to that in the LLL for $\nu = 1/2$. IT was therefore a considerable surprise when it was discovered in 1987 that a QH plateau occurs, with all the standard characteristics, at $\nu = 5/2$. This plateau seems to be quite robust, with a gap Δ which in the highest-mobility samples reaches $\sim 500\,\text{mK}$; at temperatures $\ll \Delta$ the longitudinal resistance vanishes within exponential accuracy, and the Hall conductance appears to be quantized at $(5/2)\,e^2/h$ to high accuracy, thereby excluding the possibility that the plateau is a case of the standard FQHE with $\nu = 32/13$ or $33/13$. So it really does look like a "proper" FQHE state.

Before examining the $\nu = 5/2$ state in detail, let's digress for a moment to survey the general behavior of a 2D electron gas in a strong magnetic field as a function of the filling factor $\nu \equiv nh/eB$ ($n \equiv$ areal density). We recall that in a simple case such as GaAs heterostructures where there are no complications due to valley degeneracy etc., the highest Landau level occupied (call it N) is the integer part of $\nu/2$; thus, in particular, occupation of the LLL (only) corresponds to $0 < \nu \leqslant 2$, while $N = 1$ corresponds to the range $2 < \nu \leqslant 4$.

It turns out that within the LLL, i.e. for $\nu \leqslant 2$, the system behaves much as predicted by the theory of lectures 16-20; in particular, plateaux with $\Sigma_{xy} = \nu e^2/h$ are always centered around the corresponding odd-denominator filling fractions ν, a "Fermi-liquid-like" state occurs at the even-denominator filling factor $\nu = 1/2$ (and probably also at $\nu = 3/2$, $3/4$ and $1/4$, though these cases have been less investigated). Moreover, when the system is subjected to a tilted magnetic field, the only effect seems to be to replace the total field by its component perpendicular to the plane of the 2D electron

[1] A comprehensive and up-to-date survey of the experimental aspects is given by R. Willett, Reps. Prog. Phys. **76**, 076501 (2013)

[2] In single-layer graphene, it is believed that a QH plateau would occur at $\nu = n + 1/2$ if both the spin and valley degeneracies were split, but this has to do with the special nature of the Dirac spectrum and is best regarded as a variant of the IQHE.

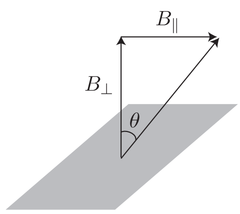
gas, indicating that any orbital[3] effects of the parallel component are negligible.

Things are very different in the range $\nu > 2(N \geqslant 1)$: see Willett, ref cit., section 3.[4] In the first place, one finds so-called "reentrant" behavior, i.e. there are plateaux corresponding to **integral** values of $\Sigma_{xy}/(e^2/h)$, but occurring around **nonintegral** values of the filling factor ν (see Willett, ref. cit., fig. 5); these are usually understood as indicating some kind of phase separation, e.g. into a "striped" phase. A second unexpected feature which appears to be restricted to the range $\nu > 4(N \geqslant 2)$ is that even in a perpendicular field, the (in-plane) resistivity close to half-filling ($\nu \cong n + 1/2$) shows **very strong anisotropy**, by a factor up to ~3,000; see Willett, ref. cit. fig. 4. This effect, while it is very sensitive to a parallel component of the magnetic field[5], which in certain circumstances can actually reverse the axis of the anisotropy, appears to be intrinsic to the sample in zero parallel field – a puzzling observation, since GaAs has the cubic ZnS structure and should ideally be completely isotropic in the xy-plane; it strongly suggests the formation of some kind of "stripes" whose orientation is fixed by small symmetry-breaking effects.

In retrospect it is perhaps not so surprising that the behavior for $N \geqslant 1$) (and particularly for $N \geqslant 2$) is more complicated than that of the LLL: In the latter case, if we adopt the symmetric gauge, the radial dependence of the single-electron wave functions has a simple Gaussian form, while for $N \geqslant 1$ it is a more complicated Hermite polynomial with zeroes; thus the Coulomb interaction (whose Fock term, we recall, is attractive for parallel spins) may be able to organize rearrangements which are not available in the LLL. Indeed, theorectical work predating the experiments anticipated such rearrangements.

While for $N \geqslant 2(\nu > 4)$ no standard FQHE has to date been seen, the second LL ($N = 1, 2 < \nu \leqslant 4$) shows a complicated pattern: along with the "reentrant" quantum Hall effect described above, there appear several plateaux which appear prima facie to have the "standard" FQHE structure, not only as observed at $\nu = 5/2$, but also at $2 + n/3$ and $2 + n/5$ (see Willett, ref. cit., fig 5). It appears that the second LL is in some sense a "battleground" between the traditional "composite-fermion" behavior of the LLL and the Coulomb effects which appear to dominate for $N \geqslant 2$.

With this in mind, let us now focus specifically on the $\nu = 5/2$ state. We first review some of its basic properties:



1. Robustness: both plateau in R_{H} and the zero of R_{xx} extend over a range ~ 0.1 in ν. (in the highest-mobility samples)[6]

2. Excitation gap Δ: this is measured by fitting R_{xx} to $R_{xx} \sim$ const. $\exp -\Delta/T$. Δ appears to be a strong ($\sim$ exponential) function of disorder: the highest measured value to date $\sim 0.45\,\mathrm{K}$, and extrapolation to zero disorder ($\mu \to \infty$) gives $\Delta \sim 0.6\,K$ ($\sim 0.006V_c$) ($V_c = e^2/4\pi\epsilon\epsilon_0 l_{\mathrm{M}}$) (Note: V_c is the natural unit to measure qp gap, as $\Delta_{\mathrm{qp}} \equiv 0$ for the FQHE if V_c is neglected).

3. Magnetic field dependence: If $B_\perp$ (hence ν) is held constant, and $B_\parallel$ is varied, Δ_{qp}

[3] Of course the Zeeman splitting remains proportional to the **total** field.

[4] It should be noted that much of the behavior described occurs only at $T \lesssim 50mK$, so that the data is quite recent.

[5] Lilly, et al., PRL **83**, 824 (1999)

[6] See fig. 1 of Xia et al., PRL **93**, 176809 (2004).

decreases linearly[7] with $B_{\|}$, extrapolating to 0 at $B_{\|} = 1.5-2.5\,\mathrm{T}$. In the same geometry, $\nu = 7/2$ FQHE shows similar behavior, but for $\nu = 7/3$ ("Laughlin" state) Δ_{qp} *increases* with $B_{\|}$.

4. At least to date, the only system in which the $\nu = 5/2$ QH plateau has been seen is GaAs heterostructures: here typical parameters are $n_s \sim 1 - 3 \times 10^{11}\,\mathrm{cm}^{-2}$ (so $B \sim 2 - 6\,\mathrm{T}$), $\mu \sim 3 \times 10^7\,\mathrm{cm}^2/\mathrm{V\,sec}$, $T \sim 5 - 100\,\mathrm{mK}$.

In trying to understand what is going on at $\nu = 5/2$ it seems very natural to make (as we have above) the default assumption that the LLL is filled both for $\uparrow$ and $\downarrow$ spin states; then prima facie the behavior in the $n = 1, \uparrow$ spin LL should be identical to that in the $n = 0, \uparrow$ spin LL for $\nu = 1/2$. However, this need not necessarily be the case, because as already observed in view of the different behavior of the wave functions for $n = 0$ and $n = 1$ the relevant matrix elements of the Coulomb interaction could be appreciably different in the two LL's. An immediate question which arises then is: given that the LLL is completely filled for both spins, i.e. unpolarized, is the $n = 1$ LL spin-polarized or not? As we shall see below, the answer to this question is crucial to the identification of the nature of the wave function and thus to the possibility of using the $\nu = 5/2$ FQHE for TQC; unfortunately, it has not so far proved possible to determine the answer experimentally. Originally, it was found that the plateau is suppressed by a substantial magnetic field component parallel to the plane, and the most obvious explanation is that the relevant state is a spin singlet (hence energetically disadvantaged by the Zeeman field). However, subsequent experiments, combined with numerical theory, tend to suggest that the effect is actually of orbital origin; because the parallel component of the magnetic field affects the motion *perpendicular* to the plane, it can change the relevant Coulomb matrix elements. For the moment, let us assume that the $\nu = 5/2$ state seen in GaAs heterostructures is in fact fully spin polarized (we will return to this question below) and ask what is its nature? In particular, why does a QH system at $\nu = 5/2$ not behave, as it seems to at $\nu = 1/2$, as a "disguised" Fermi liquid (of composite fermions)?

A relevant question is: What do we know about the possible instabilities of a Fermi liquid? There are of course a great many, but most of them, such as crystallization, tend to occur either not at all or at temperatures comparable to the Fermi temperature, which for GaAs heterostructures at $n = 10^{11}\,\mathrm{cm}^{-2}$ is $\sim 30\,\mathrm{K}$. The obvious instability which occurs for arbitrary weak interactions of the right sign and thus at arbitrary low temperatures is *Cooper pairing*, which of course in a system of real particles leads to superconductivity (if charged) or superfluidity (if neutral). This consideration led Moore and Read[8] to conjecture that

the $\nu = 5/2$ QH plateau corresponds to a Cooper-paired state of composite fermions.

If this is true, then it is generally believed that the elementary excitations will be non-abelian (Ising) anyons, which is what makes this possibility so interesting in the context

[7]Dean et al., PRL **101** 186806 (2008).

[8]Nuc. Phys. **360**, 362 (1991).

of TQC.

Let's try to make this hypothesis a bit more quantitative. According to the composite-fermion hypothesis, the correct groundstate for a given value of $\nu = k/(2nk \pm 1)$ is given by taking the non-Gaussian part of the wave function of electrons at $\nu = k$, multiplying by $(z_i - z_j)^{2n}$ ("adding $2n$ flux quanta to each electron") and readjusting the Gaussian part so that the magnetic length l_{M} which comes in refers to the actual magnetic field B (not B^*). Now, consider the case $\nu = 1/2$. This corresponds to the choice $k \to \infty$, $n = 1$; since in this limit $B^* = 0$, the wave function of a set of *noninteracting* electrons is just the filled Fermi sea for given value of n_s and complete spin polarization $|\mathrm{FS}\rangle$. Consequently, according to the above prescription we should have apart from normalization

$$\Psi^{(\mathrm{normal})}_{\nu=1/2} = \prod_{ij}(z_i - z_j)^2 \exp - \sum_i |z_i|^2/4l_{\mathrm{M}}^2 \,|\mathrm{FS}\rangle \tag{1}$$

where the $|\mathrm{FS}\rangle$ component guarantees the correct antisymmetry under the exchange $i \leftrightarrow j$. The assumption which seems to be implicit in much of the theoretical literature is that a similar expression, multiplied by the appropriate Slater determinants for the completely filled (spin singlet) LLL wave function, would be adequate also for the $\nu = 5/2$ state were it not for the effect of interactions. In the following I will not write out the part of the wave function which refers to the LLL explicitly.

So, if eqn. (1) is the correct representation of a normal Fermi sea of composite fermions, what is the corresponding representation for a BCS-paired state? The obvious answer is

$$\Psi^{(\mathrm{paired})}_{\nu=1/2} = \prod_{ij}(z_i - z_j)^2 \exp - \sum_i |z_i|^2/4l_{\mathrm{M}}^2 \,|\mathrm{BCS}\rangle \tag{2}$$

where $|\mathrm{BCS}\rangle$ is the Cooper-paired state of weakly interacting fermions at the relevant density[9]. Now, on our assumption that the $\nu = 5/2$ state (as well as the $\nu = 1/2$ state) is completely spin-polarized, the Fermi antisymmetry requires that the pairing takes place in a state of odd relative orbital angular momentum l, and the default option is $l = 1$ (p-state). Moreover, the state must be two-dimensional.[10] This still does not specify the state uniquely; for example, the order parameter could be of the form Ak_x (or Ak_y), which breaks rotational invariance but not time-reversal invariance. However, our general experience with BCS pairing suggests that in a rotationally invariant system it is usually energetically advantageous to make the energy gap (which is proportional to the modulus of the order parameter) as uniform as possible over the Fermi surface. This suggests that we should choose for the OP

$$\Delta(\mathbf{k}) = \Delta_0(k_x \pm ik_y) \tag{3}$$

so that even though the magnetic field B^* acting on the composite fermions is zero, their state still breaks time reversal invariance. In the literature a state of the form (3)

[9] From now on we shall always implicity assume that the total number of particles N is even.

[10] Of course, as we have seen in lecture 9, no true superfluid ODLRO can survive in 2D. However, we may assume, at least for the moment, that we are below the KT transition, so that the lack of ODLRO does not affect the qualitative behavior.

is referred to as a "$p + ip$" state: note that the "energy gap" is independent of $\mathbf{k}$ and equal to Δ_0.

The crucial question, now, is: What is the explicit form of the groundstate wave function $|\mathrm{BCS}\rangle$ which corresponds to the choice (3)? Actually, as we shall see in the next lecture, the answer does not seem to be unique in the thermodynamic limit (a fact which has not been widely appreciated in the QH literature). However, there is a particular answer which has been widely given in the literature on superfluid ^{3}He and other (non-FQH) condensed matter systems, namely that the wave function is of the standard BCS form, with the coefficients $u_\mathbf{k}$ and $v_\mathbf{k}$ having the right angular dependence for a $p + ip$ state. This is, explicitly,

$$|\mathrm{BCS}\rangle = |\mathrm{BCS}\rangle_{\mathrm{standard}} \equiv \prod_\mathbf{k} (u_\mathbf{k} + v_\mathbf{k} a^\dagger_\mathbf{k} a^\dagger_{-\mathbf{k}})|0\rangle \tag{4}$$

where $|0\rangle$ is the physical vacuum, the spin suffix $\uparrow$ is omitted for simplicity, and the coefficients $u_\mathbf{k}$ and $v_\mathbf{k}$ are given by

$$\begin{gathered} u_\mathbf{k} = \frac{1}{\sqrt{2}}(1+\epsilon_\mathbf{k}/E_\mathbf{k}), \quad v_\mathbf{k} = \frac{1}{\sqrt{2}}(1 - \epsilon_\mathbf{k}/E_\mathbf{k}) \exp i\phi_\mathbf{k} \\ E_\mathbf{k} \equiv \sqrt{(\epsilon_\mathbf{k} - \mu)^2 + \Delta_0^2} \end{gathered} \tag{5}$$

with μ the chemical potential. It is easily verified that the expectation value of the total orbital angular momentum L in the state described by $|\mathrm{BCS}\rangle_{\mathrm{standard}}$ is $N\hbar/2$.

Should we simply insert $|\mathrm{BCS}\rangle_{\mathrm{standard}}$ in the conjecture (2) for the groundstate of the $\nu = 5/2$ QH system? There is a problem here, since we do not know a priori the value of the effective Coulomb matrix elements for the composite-fermion states and hence cannot calculate the gap magnitude Δ_0. However, since the order of magnitude of the Coulomb energy ($e^2/4\pi\epsilon\epsilon_0 r_0$, where $r_0 \sim n_s^{-1/2}$) is quite comparable to the Fermi energy and may in fact be larger, it seems reasonable to suppose that Δ_0 is of order ϵ_F and thus that the pair radius ξ of the pairs in the state $|\mathrm{BCS}\rangle_{\mathrm{standard}}$ is of order of the interparticle spacing (or the magnetic length, which for $\nu \sim 1$ is essentially the same thing). But there is little reason to believe that the composite-fermion idea works quantitatively on this kind of scale. Consequently, it may seem sensible to insert in (2) not the full real-space wave function derived from (4), but only the form which the latter takes at long distances ($|\mathbf{r}_i - \mathbf{r}_j| \gg k_\mathrm{F}^{-1}, \xi$). As we shall see in the next lecture, this has the form of the "Pfaffian"

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \equiv \frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\frac{1}{z_5 - z_6}\ldots - \frac{1}{z_1 - z_3}\frac{1}{z_2 - z_4}\frac{1}{z_5 - z_6}\ldots + \ldots - \ldots \tag{6}$$

i.e. it is the completely antisymmetrized version of the expression $\prod^N_{\substack{i=1\\j=i+1}} \left(\frac{1}{z_i - z_j}\right)$.

Thus, finally, the ansatz of Moore and Read for the $\nu = 5/2$ QH state is up to normalization (presumably omitting the filled LLL)

$$\boxed{\Psi_{\mathrm{MR}}\{z_i\} = \prod_{i<j}(z_i - z_j)^2 \,\mathrm{Pf}\, \frac{1}{(z_i - z_j)} \exp - \sum_i |z_i|^2/4l_\mathrm{M}^2} \tag{7}$$

and it is on this conjectured form that most of the work on the possible implementation of TQC in this system has been based. Note that apart from the Pfaffian factor, Ψ_{MR} is just the Laughlin factor for $\nu = 1/2$; however, the Pfaffian factor is of course essential to give the correct antisymmetry.

Despite this, it is important to note that when we are counting powers of z_i, this factor only adds one for each added pair of particles, whereas the Laughlin factor adds $4N$; thus the former is negligible in this context in the thermodynamic limit.

Although the ansatz (7) is an informed guess, it can be made plausible by two considerations:

1. It is the exact groundstate of the artificial Hamiltonian

$$\hat{H} = +V_0 \sum_{ijk} \delta^{(2)}(\mathbf{r}_i - \mathbf{r}_j)\delta^{(2)}(\mathbf{r}_j - \mathbf{r}_k) \quad \big(\delta^{(2)}(\mathbf{r}) \equiv \delta(x)\delta(y)\big) \tag{8}$$

2. Numerical studies show that it has a substantial overlap with the exact groundstate of some rather more realistic model Hamiltonians.

It is sometimes said in the literature that the MR state (7) is the exact analog of the superfluid A_1 state of liquid ^{3}He. This is not strictly true, since as well as the paired up-spin component ^{3}He-A_1, also has an unpaired down-spin component. It would be true for (hypothetical) totally spin-polarized ^{3}He-A_1, with no down-spin component. Note by the way that while the $\nu = 1/2$ state would appear to be symmetric with respect to particles and holes, the Hamiltonian (8) breaks the particle-hole symmetry, and since the MR state is an exact eigenfunction of it, it must do the same.

Let's briefly review some other possible identifications of the $\nu = 5/2$ QH state.[11]

1. The (331) state

   Like the MR state, this is a triplet-paired state of the composite fermions; however, unlike that state (which corresponds to $S = 1$, $S_z = 1$) this one corresponds to $S = 1$, $S_z = 0$ and hence has no net spin polarization in any direction. The explicit form of this state in terms of the electron coordinates is

$$\Psi = \sum_{\{\sigma_i\}:\sum_i \sigma_i = 0} \prod_{\substack{i<j \\ \sigma_i = \sigma_j}} (z_i - z_j)^3 \prod_{\substack{i<j \\ \sigma_i \neq \sigma_j}} (z_i - z_j)^1 \ \exp - \sum_i |z_i|^2/4l_{\text{M}}^2 \tag{9}$$

   In words: the correlation of any two parallel-spin electrons vanishes as r^3, but correlation of any two anti-parallel spin electrons vanishes only as r.

   The 331 state *is* the exact analog of the A phase of liquid ^{3}He (which has no net Cooper pair polarization).

   In terms of the original electron coordinates, Ψ_{MR} and Ψ_{331} look totally different, and in particular appear to have a different topology. But when expressed in terms

[11] The most readable account of this subject I know is T-L. Ho, PRL **75**, 1186 (1995).

of the composite fermions (see Ho, ref. cit.), it turns out that the only difference is in the spin wave function $\chi_{\mu\nu}$!

$$\chi_{\mathrm{MR}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \chi_{331} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{10}$$

Ho exploits this fact to show that Ψ_{331} can be deformed continuously into Ψ_{MR} without changing "total" $g(\mathbf{r}_{12}) \equiv \langle \rho(\mathbf{r}_1)\rho(\mathbf{r}_2)\rangle$, hence without changing V_c. (Cf.: non-metastability of circulating state of spin-1/2 BEC in annulus). The apparent paradox thereby arising is discussed in detail in section 11 of the paper by Nayak and Wilczek (ref. (14) below)

2. The "anti-Pfaffian" state.[12]

   If one could neglect completely LL mixing, particle-hole conjugation is an exact symmetry for a half-filled LL. But Ψ_{MR} breaks this symmetry, since it is the exact GS of a Hamiltonian which is *not* particle-hole symmetric. So there must exist an "anti-Pfaffian" state (Ψ_{AP}) which is the particle-hole conjugate of Ψ_{MR}. For exact particle-hole symmetry it must be degenerate with Ψ_{MR}, but LL mixing could stabilize either Ψ_{AP} (or Ψ_{MR}). To the best of my knowledge, no-one has formulated the AP wavefunction explicitly in terms of composite fermion coordinates, so one cannot immediately compare it with the MR or 331 states. However, studies of the edge states using bosonization predict different values for the ("universal") thermal conductance, etc. (see below).

3. Other possible identifications.

   In the literature there have been yet other suggestions for the nature of the $\nu = 5/2$ QH state: the "$K = 8$" state, the "$U(1) \times SU_2(2)$" state and others. I will not go into the details here

Let's now turn to the important but rather confusing issue of the charge and statistics of the MR states and its competitors. I will try to give a plausible argument only for the MR and (331) states, and just quote the results for the others.

For a first pass, let us look at the elementary excitations of a $(p + ip)$ 2D Fermi superfluid. These are (mostly, cf. lecture 27) of two types: simple fermionic BCS quasi-particles, with a minimum excitation energy equal to the gap Δ_0, and vortex-antivortex pairs, whose characteristic energy is strongly temperature-dependent as discussed in lecture 10. Now for a strictly 2D system with $\Delta \sim \epsilon_{\mathrm{F}}$, the fraction of excited quasiparticles at T_{KT} is fairly small (Problem), so let us focus on the vortex-antivortex pairs. Suppose we want to create a vortex at the origin. In a BCS superfluid the way to do this is simply to multiply the wave function by a factor of the form $\prod_{i=1}^{N} f(|z_i|)\exp(i\phi_i/2)$, where $\phi_i \equiv \arg z_i$ and $f(|z_i|)$ is some function which tends to zero as $|z_i| \to 0$. The factor of 1/2 in the phase corresponds to the well-known fact that in a neutral superfluid such as ^{3}He the vorticity is quantized in units of h/m_{p} rather than h/m, where $m_{\mathrm{p}} \equiv 2m$ is the mass

[12] Levin et al., PRL **99**, 236806 (2007): Lee et al., ibid. 236807.

of a Cooper pair. The simplest form of $f(|z_i|)$ which preserves the analyticity of the wave function up to a cut is $f(|z_i|) = |z_i|$; thus, a vortex at the origin might be created simply by multiplying the groundstate wave function by $\prod_{i=1}^{N} z_i^{1/2}$, and correspondingly a vortex at the position specified by the complex variable $x + iy \equiv \eta_0$ would then be created by application of $\prod_{i=1}^{N}(z_i - \eta_0)^{1/2}$. Arguing along these lines, we would conclude that a plausible ansatz for a single quasiparticle (actually quasihole) in a system whose GS is described by the MR wave function is

$$\Psi_{\rm qh} = \prod_{i=1}^{N}(z_i - \eta_0)^{1/2}\Psi_{\rm MR}\{z_i\} \tag{11}$$

Note that a simple power-counting argument (cf. below) would then show that the quasiparticle charge is e/4.

Whether (11) is correct or not[13] may be a matter of theology, since it actually turns out that it is impossible to create a single isolated vortex in a $(p + ip)$ Fermi superfluid; for essentially topological reasons one must have either an even number of vortices, or an edge state which plays the same role as a vortex. So the physically meaningful question in the case of the $\nu = 5/2$ QH state is, what is the correct form of the wave function for *two* quasiholes? The ansatz made by Moore and Read (their eqn. (5.9), for the special case $q = 2$, i.e. $\nu = 1/2$) takes the rather complicated-looking form.

$$\Psi_{pair}(z_1 z_2..z_N : \eta_1, \eta_2) = (\eta_1 - \eta_2)^{-1/4}\times$$

$$\sum_{\sigma\epsilon S_n} sgn\sigma \prod_{k=1}^{N/2} \frac{[(z_{\sigma(2k-1)} - \eta_1)(z_{\sigma(2k)} - \eta_2) + (\eta_1 \rightleftarrows \eta_2)}{(z_{\sigma_{(1)}} - z_{\sigma(2)})....(z_{\sigma(N-1)} - z_{\sigma(N)}).} \tag{12}$$

$$\times \prod_{i<j}(z_i - z_j)^2 \exp(-\frac{1}{4}[|z_i|^2) \leftarrow [\frac{1}{4}\sum_i |z_i|^2] \qquad \leftarrow [\frac{1}{4}\sum_i |z_i|^2$$

where η_1, η_2 denote the quasihole positions and σ is any permutation of the z_i's, and sgn σ is the parity of the number of interchanges involved. Several things about eqn. (12) should be noted:

1. It is correctly antisymmetric in the z_i's but (apart from the prefactor, on which more below) *symmetric* in the η_i
2. It is (unlike (11)) analytic in the z_i, but *cannot* in general be written in the form of $f(z_i, z_j) \times \Psi_{GS}$
3. The product over k runs only up to $N/2$, not up to N. Hence the number of extra powers of z_i by comparison with the groundstate is N, not $2N$ (remember that the denominator introduces only a power 2, which is negligible in the thermodynamic limit).

[13] In the literature it is usually stated that the single quasihole creation operator is $\prod_{i=1}^{N}(z_i - \eta_0)$ as for Laughlin states, but it is not clear that this statement has any real meaning.

4. It may be verified that if we create *four* of the pairs described by eqn. (12) (with an appropriate generalization of it, see below) at the same two points η_1, η_2 and follow the procedure employed in lecture 18 for the Laughlin states, we recover the groundstate (eqn. (7) with 2 extra electrons. However, it is essential to appreciate that *this is only true because we have explicitly included the prefactor* $(\eta_1 - \eta_2)^{-1/4}$, which is thus seen to be an essential element in the ansatz (12).

From the above considerations it is straightforward to infer the charge and statistics of the quasiparticles of the MR state as generated by eqn. (12). The charge follows intuitively from power-counting according to (3) above: Since each pair of quasiholes adds a power N, while addition of an extra pair of electrons requires $4N$ extra powers, the charge e^* of a single quasihole must be $e/4$. A more rigorous demonstration of this result follows from (4). Thus

$$\boxed{e^*_{MR} = e/4} \tag{13}$$

As we will see below, this result is not unique to the MR state.

What of the quasihole statistics? Suppose we encircle the (stationary) quasihole 2 by quasihole 1, in the usual adiabatic way. Then the factor $z_i - \eta_1$ in the numerator give rise, just as in the Laughlin case, to a phase factor $\exp 2\pi i\nu = -1$; so at first sight the quasiholes would appear to behave as simple fermions. However, we need to remember the overall prefactor $(\eta_1 - \eta_2)^{-1/4}$ (which we recall had to be there to guarantee the correct cancellation of 4 quasiholes by 1 electron). When this is taken into account the correct "encirclement phase" is $3\pi/2$, which is equivalent to $(-)\pi/2$, so the exchange phase is $\pi/4$.

However, an exchange phase of $\pi/4$ is not in itself inconsistent with Abelian statistics, so let us now examine this point. Consider a 4-qh state (i.e. 2 qh pairs).[14] Start with the somewhat artificial case of $N = 2$, then a possible wave function. is

$$\Psi_{4\,\mathrm{qh}} = \Psi^{(\mathrm{L})}(z_1 - z_2)^{-1} \times \{(z_1 - \eta_1)(z_1 - \eta_2)(z_2 - \eta_3)(z_2 - \eta_4) + (z_1 \leftrightarrow z_2)\} \equiv (12)(34) \tag{14}$$

But at first sight there are two other possibilities, namely (13)(24) and (14)(23). However, we now note the identity

$$(12)(34) - (13)(24) = (z_1 - z_2)^2(\eta_1 - \eta_4)(\eta_2 - \eta_3) \tag{15}$$

from which it follows that

$$(12)(34)(\eta_1 - \eta_2)(\eta_3 - \eta_4) + (13)(24)(\eta_1 - \eta_3)(\eta_2 - \eta_4) + (14)(23)(\eta_1 - \eta_4)(\eta_2 - \eta_3) = 0 \tag{16}$$

i.e. *only 2 linearly independent functions.* This result also holds for the generalization for $N > 2$:

$$\Psi_{4\,\mathrm{qh}} = \Psi_N^{(\mathrm{L})}\mathrm{Pf}\ \left\{\frac{(13)(24)}{z_1 - z_2} - \frac{(13)(24)}{z_3 - z_4}\dots\right\} \tag{17}$$

[14]Nayak and Wilczek, Nuc. Phys. B **479**, 529 (1996).

and it can be generalized to the case of $2n$ quasiholes :

$$\text{for } 2n \text{ quasiholes, } 2^{n-1} \text{ linearly independent states}$$

NW exhibit an explicit form of a possible choice[15] of basis states for $n = 2$:

$$\begin{aligned}
\Psi^{(0)}_{4\,\mathrm{qh}} &= \frac{(\eta_{13}\eta_{24})^{1/4}}{(1+\sqrt{1-x})^{1/2}} \left(\Psi_{(13)(24)} + \sqrt{1-x}\,\Psi_{(14)(23)}\right) \\
\Psi^{(1/2)}_{4\,\mathrm{qh}} &= \frac{(\eta_{13}\eta_{24})^{1/4}}{(1-\sqrt{1-x})^{1/2}} \left(\Psi_{(13)(24)} - \sqrt{1-x}\,\Psi_{(14)(23)}\right) \\
&(?) \quad x \equiv \frac{\eta_{12}\eta_{34}}{\eta_{13}\eta_{24}} \qquad \eta_{12} \equiv \eta_1 - \eta_2, \quad \text{etc.}
\end{aligned} \tag{18}$$

Let's take $|x| \ll 1$ i.e.

so that

$$\begin{aligned}
\Psi^{(0)}_{4\,\mathrm{qh}} &= 2^{-1/2}(\eta_{13}\eta_{24})^{1/4} \left(\Psi_{(13)(24)} + \Psi_{(14)(23)}\right) \\
\Psi^{(1/2)}_{4\,\mathrm{qh}} &= 2^{-1/2}(\eta_{13}\eta_{24})^{1/4} \left(\Psi_{(13)(24)} - \Psi_{(14)(23)}\right)
\end{aligned} \tag{19}$$

Then it is clear that interchange of 1 and 3 (or 2 and 4) affects only the prefactor, so gives a phase factor $\exp i\pi/4$. It would thus be natural to take the charge e^* of a quasihole to be $e/4$, as we already deduced.

However, interchange of (e.g.) 2 and 3 gives a nontrivial rotation[16] in the space of $\Psi^{(0)}_0$ and $\Psi^{(1/2)}$. Thus, the states $\Psi^{(0)}_{4\,\mathrm{qh}}$ and $\Psi^{(1/2)}_{4\,\mathrm{qh}}$ can in principle be used as the basis for a qubit. More generally,

$$2n \textbf{ anyons} \rightarrow n \text{ qubits}$$

At this point let us note that the (331) state is much less exotic. In fact, it is essentially a product of simple Laughlin states of each spin population corresponding to $\nu = 1/4$, with the only difference being that to get the statistics right one of the 4 powers of $(z_i - z_j)$ has to involve correlation with the opposite spin population. When viewed in this way, it is clear that simple power-counting gives the excitations a fractional charge $e^* = e/4$ just as in the Pfaffian case; however, encirclement of one quasiparticle by another (whether of the same of opposite spin species) proceeds just as in the Laughlin case and gives (only) a phase factor $\exp\, i\pi/4$. Thus the statistics of the (331) phase is abelian.

[15]That the explicit phase factor is in the numerator rather than, as in eqn. (12), in the denominator merely changes the encirclement phase from $3\pi/2$ to $\pi/2$ and is irrelevant to the physical results.

[16]Confirmed by numerical calculations: Tserkovnyak and Simon, PRL **90**, 016802 (2003).

We now turn to the question: How do we tell experimentally whether the observed $\nu = 5/2$ QH state is indeed the MR state (as numerical studies tend to suggest) or is one of the competing states ((331), antiPfaffian etc.)? It turns out that all the suggested identifications predict[17] that the effective charge e^* is $e/4$, but they predict different values for the "Coulomb exponent" g which controls some of the properties associated with edge states, e.g. temperature-dependence of tunnelling characteristics.

Ansatz	Spin-polarized?	e^*	g	abelian/nonabelian
MR	yes	$e/4$	0.25	nonabelian
AP	yes	$e/4$	0.5	nonabelian
331	no	$e/4$	0.375	abelian
$K = 8$	yes(?)	$e/4$	0.125	abelian
$U(1) \times SU_2(2)$	yes	$e/4$	0.5	nonabelian

Experiments designed to identify the nature of the $\nu = 5/2$ FQHE state

1. Is it spin-polarized?

This question is discussed in some detail in section 4 of the Willett review. As we have seen, the early observation that a sufficiently large in-plane magnetic field destroys the $\nu = 5/2$ QH plateau was originally taken as evidence for spin singlet pairing (which would be destroyed via the Zeeman effect); however subsequent experiments have suggested that the effect of the field is more likely to be via its orbital interaction, so this argument is no longer widely believed. Other experiments designed to determine the spin polarization include electrically detected NMR, polarization-sensitive photoluminescence and acoustic resonance measurements at temperatures above that for formation of the QHE plateau (which in effect measure the Fermi wave vector and thus, knowing the areal density, the spin polarization). Willett's conclusion is that the evidence from these experiments for a completly spin-polarized state, while suggestive, is not conclusive.

2. Shot-noise experiments

1. Dolev et al., (Weizmann Institute) Nature **452**, 829 (2008). Sample: GaAs-AlGaAs heterostructure.

$n \sim 3 \times 10^{11}\,\mathrm{cm}^{-2}$
$\mu \sim 3 \times 10^{7}\,\mathrm{cm}^2/\mathrm{V\,sec}$
$T \sim 10\,\mathrm{mK}$

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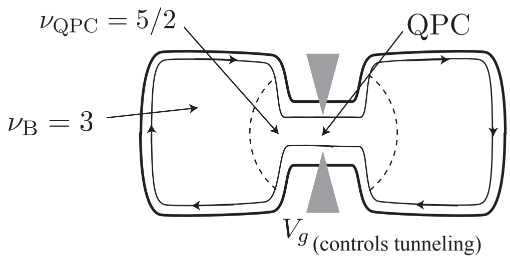
[17] I suspect this follows from rather general topological considerations, see below.



Measure: shot noise associated with tunnelling across QPC (quantum point contact). Theoretical prediction:

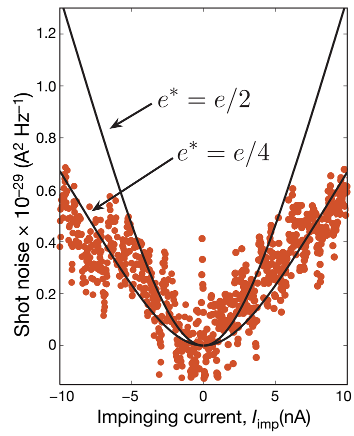
$$S_I = 2e^* V \Delta g_i t_i (1 - t_i) \left[\coth\left(\frac{e^* V}{2k_{\mathrm{B}}T} \right) - \frac{2k_{\mathrm{B}}T}{e^* V} \right] + 4k_{\mathrm{B}}Tg \tag{20}$$

Note that: (a) for $k_{\mathrm{B}}T \gg e^* V$, the [ ] goes to 0 so no information on e^*, (b) for $k_{\mathrm{B}}T \ll e^* V$, $S_I = 2e^* V \Delta g_i t_i (1 - t_i)$, where $\Delta g_i = (\nu_i - \nu_{i-1}) e^2/h$ with $t_i \equiv$ tunneling over edge between i and $i - 1 \Rightarrow$ must know Δg_i, t_i (which may depend on I_{imp}).

Typical data: (note theoretical curves need knowledge of g_i, t_i - taken from $\nu = 3$ measurements (?)).

Conclusion: $e^* = e/4$ with small/zero $e/2$ contamination, no conclusion about g.

(adapted from Dolev et al.)



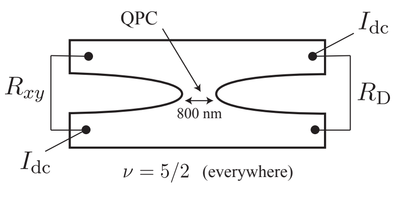
2. Radu et al. (Harvard-MIT-Lucent), Science **320**, 899 (May 2008).
Sample: GaAs-AlGaAs heterostructure.

$n \sim 2 - 6 \times 10^{11}\,\mathrm{cm}^{-2}$

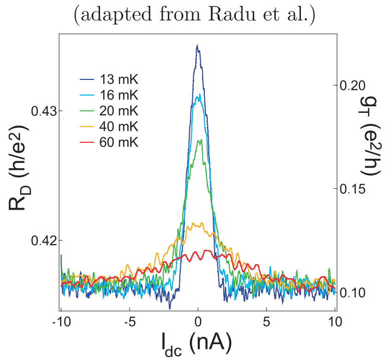
$\mu \sim 2 \times 10^7\,\mathrm{cm}^2/\mathrm{V\,sec}$

$T \sim 13 - 60\,\mathrm{mK}$

Measure: V_{D} (i.e. R_{D}) and V_{xy} (i.e. R_{xy}) at fixed I_{dc} and V_g, infer the tunneling conductance of the QPC by $g_{\mathrm{T}} = (R_{\mathrm{D}} - R_{xy})/R_{xy}^2$. Plot R_{D} ($\propto g_{\mathrm{T}} + \mathrm{const}$) as a function of I_{dc}.



(adapted from Radu et al.)



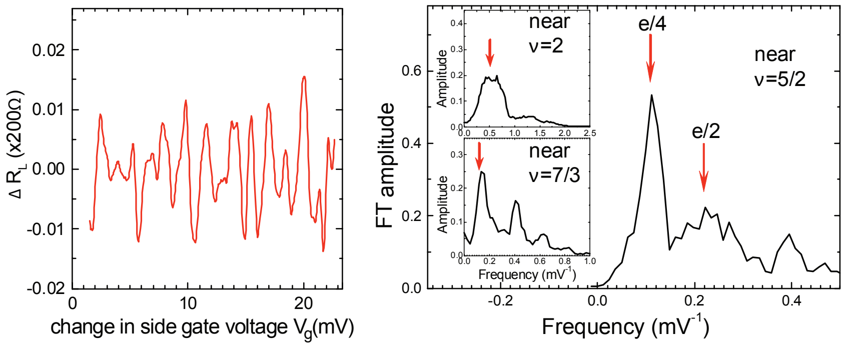
Typical data are shown on the graph above. Fit data to (weak-tunneling) expression

$$g_{\mathrm{T}} = AT^{2g-2}F(g, e^*V/k_{\mathrm{B}}T) \quad V \equiv I_{\mathrm{dc}}R_{xy} \tag{21}$$

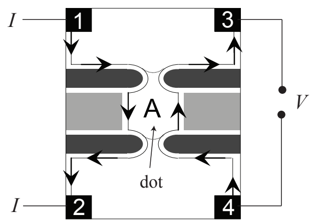
where F is a known function. Four fitting parameters: A, R^∞, e^*, g. Best fit: $e^* = 0.17$, $g = 0.35$ i.e. AP or $U(1) \times SU_2(2)$ ($e^* = 0.25$, $g = 0.5$). Data barely consistent with (331) state ($g = 0.375$), probably *inconsistent* with MR ($g = 0.25$).

3. Willett et al., (Lucent), PNAS **106**, 8853 (2 June 09). Sample: GaAs-AlGaAs heterostructure.

(adapted from Willett et al.)



(adapted from Willett et al.)



$$n \sim 4 \times 10^{11}\,\mathrm{cm}^{-2}$$
$$\mu \sim 2 - 5 \times 10^{7}\,\mathrm{cm}^2/\mathrm{V\,sec}$$
$$T \sim 25 - 150\,\mathrm{mK}$$

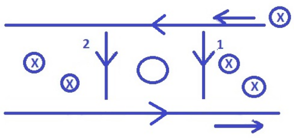
Measure: dependence of R_L ($\equiv V/I$) on magnetic field B and gate voltage V_g . By calibrating with nearby well-understood QHE plateaux ($\nu = 5/3$, 2, 7/3), can infer effective area A of the dot as a function of V_g. Note effect of V_g is not primarily through charge accumulation on dot, but directly through change of area $\Rightarrow$ change of enclosed flux. Hence, should be a unique relation between period observed in B and V_g.

Conclusion: at low T, main component is $e^* = e/4$, but with an appreciable $e/2$ component. At higher T, $e/2$ dominates.

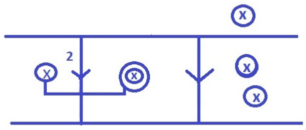
A more sophisticated and interesting version of this experiment, using the same geometry, exploits the following idea[18]:

Consider the interference between the two paths 1 and 2 available to an $(e/4)$ quasi-particle which is to be back-reflected: In fig. (a), where there are no pre-existing $e/4$ quasiparticles trapped on the dot, neither its passage down path 1 nor passage down path 2 affects the state of the existing quasiparticlas, so the two processes can interfere coherently and we expect to see the appropriate periodic dependence of the current on

[18] see e.g. Stern and Halperin, PRL **96**, 016802 (2006)



(a)



(b)

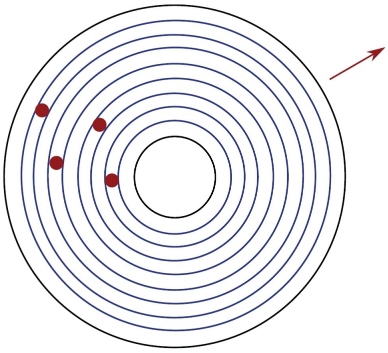
the magnetic field, gate voltage etc. applied to the dot. The same applies if there is an even number of $e/4$ quasiparticles trapped on the dot: again, the state of the pre-existing quasiparticles changes at most by a phase factor, which simply shifts that interference pattern without changing its periodicity. The situation is different when there is a single $e/4$ quasiparticle on the dot: as we have seen above, if we link that quasiparticle to one outside the interference area to form a qubit, and encircle it (but not its partner) with the incoming quasiparticle, we change the state of the qubit to an orthogonal state. Thus, if the incoming quasiparticle takes path 1, the final state Ψ_f of the pre-existing quasiparticles (or the qubits they form) is identical to their initial state Ψ_{in}, whereas if it takes path 2, Ψ_f is orthogonal to Ψ_{in} and thus to the final state for path 1. Thus, for a single $e/4$ quasiparticle on the dot, *the interference between paths 1 and 2 should be destroyed* and we should prima facie see no periodic dependence of the current on the magnetic field, voltage etc. applied to the dot. The same applies to any odd number of $e/4$ quasiparticles on the dot. Thus, if the parity of the number n_d of $e/4$ quasiparticles in the dot is even/odd, we should prima facie see/not see an oscillating dependence of the current on the dot parameters. Actually, it turns out that the situation is a bit more complicated, because two (incoming) $e/4$ quasiparticles can combine to form an $e/2$ quasiparticle, which is an abelian object: consequently, there should exist independently of the dot parity a background periodicity of the dependence on dot parameters of $e/2$. Assuming this component is small compared to the $e/4$ one, we would then predict: For even parity of n_d the periodicity in the dot parameters should be dominantly $e/4$ for odd parity $e/2$.

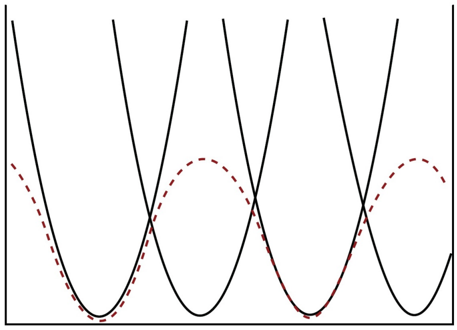
Attempts to test this prediction over the last 4 years are discussed in detail in the Willett review. To summarize, while there are certainly regions of the parameters (magnetic field, gate voltage) for which an $e/4$ oscillations are dominant, and others in which only $e/2$ appears to occur, one cannot say that it has been definitively established that these regions are discriminated by the parity of the number of $e/4$ excitations on the dot. This is very much a work in progress (cf. Willett's penultimate sentence).

What can we say generically about $\nu = 5/2$?

A). On torus, by generic Wen-Niu argument, groundstate must be at least doubly degenerate.

B). Hall effect in "wide" Corbino-disk geometry: By original Laughlin argument, $2\phi_0$ of flux must correspond to e of charge. So minimum "accessible" periodicity of F in Φ is $2\phi_0$: e.g. could have the situation as on the figure with single electron making





"adiabatic" transition.

But: $1/2 = 2/4$! So, equally plausible scenario is shown on the bottom of the page, with 2 electrons making "adiabatic" transition. This would almost certainly gives "noise" corresponding to $e/4$ in "constricted" Corbino-disk geometry.

So, $e^* = e/4$ merely indicates "pairing" and nothing more specific?

[If time permits I will briefly discuss also the $\nu = 12/5$ state, one candidate state which has anyon excitations of the Fibonacci rather than Ising type.]