Problem Sheet 3

1. "Brute-force" approach to the 1D Ising model.

   Consider a 1D chain with Hamiltonian

   $$\hat{H} = \frac{-J}{2}\sum_i \sigma_i\sigma_{i+1} \qquad \sigma_i = \pm 1. \tag{1}$$

   (a) (trivial) Write down an expression for the energy as a function of the total magnetization $M = \sum_i \sigma_i$ and the number of "kinks" k.

   (b) Find an expression for the entropy S(M,k) in the thermodynamic limit. (Hint: Use the fact that in this limit the problem essentially reduces to the number of ways one can partition the "up" and "down" spins each into k/2 groups). Hence write down an expression for the free energy F(M,k) = E(M,k) − TS(M,k).

   (c) Show that the most probable value of k is given by the equation

   $$k^2 = ((N-k)^2 - M^2)\exp - J/T \equiv k_m^2(M) \tag{2}$$

   (d) By inserting the value of k into the result of (b), find an (approximate) expression for the entropy S(M) $\cong$ $\max_k$ S(M,k), and show that in the high-temperature limit this coincides with the result of a direct calculation of S(M) $\equiv$ k_B ln W(M) (W = number of configurations). (Why is this result reasonable?)

   (e) Now add to the Hamiltonian a Zeeman term of the form

   $$\hat{H}_2 = -\mu(\sum_i \sigma_i)\mathcal{H} \tag{3}$$

   Where $\mathcal{H}$ is the external magnetic field. Find an equation similar to (2) for $\mathcal{H}$ in terms of M, k and T.

   (f) By substituting this equation into (2), find the equation of state, i.e. an equation for the magnetization M as a function of $\mathcal{H}$ and T. It should reduce to the "textbook" form.[1]

$$M = \frac{N sinh(\mu \mathcal{H}/k_B T)}{\sqrt{sinh^2(\mu \mathcal{H}/k_B T) + e^{-J/K_B T}}} \tag{4}$$

(g) Check that (4) is "physically sensible" in the limits $J \ll k_B T$ and $J \gg k_B T$, and that the system is not ferromagnetic at any nonzero T.

2. (a) (very easy) By using standard Ginzburg-Landau (mean-field) theory and the definition of the superfluid density $\rho_s(T)$, show that in the limit $T \to T_c$ ρ_s has the form for both superfluid ^{4}He and superconductors (within mean-field theory)

$$\rho_s(T) = \text{const. } \rho(1 - T/T_c) \equiv \text{const. } \rho t \tag{5}$$

where ρ is the total density, and give an argument that for a translation-invariant system the constant must be of order unity.

(b) Consider liquid ^{4}He in an annular pore of (circular) cross-section $(50\text{Å})^2$ and circumference 1 cm. (This might be a primitive model of ^{4}He in Vycor). Make a rough estimate, using the result of part (a), of the value t_0 of t ($\equiv 1 - T/T_c$) at which the system loses long-range order.

(c) The same as for (b), for a film of ^{4}He of thickness 10 Å and area 1 cm^2.

(d) Are the results qualitatively affected by the replacement of the mean-field form of $\rho_s(T)$ in part (a) by what is believed to be the true behavior in the limit $T \to T_c$, namely $\rho_s(T) \sim \rho(1 - T/T_c)^\zeta$, $\zeta \approx 2/3$?

3. Consider the "Langer-Fisher" (LF) mechanism for decay of superflow, in which a (3D) vortex ring is nucleated and (may) expand to the boundaries of the system.

(a) Using the formulae of lecture 10, determine the approximate dependence of the free energy barrier for this process on the superfluid velocity v_s.

(b) For the situation discussed in problem 2, part (b), at temperatures of the order of t_0, which process is more probable: the LF process, or a "bulk" phase slip

[1] Most standard texts on statistical mechanics derive this result by an alternative method; beware however of different definitions of J and μ.

in which a complete cross-section of the pore is turned normal?[2] Consider this question (i) for the lowest circulating state ($\kappa = 1$) (ii) for $v_s \sim 10\,\text{m/sec}$.

(c) Under the conditions of part (b), is the system "effectively" superfluid?

Solutions due by 9 a.m. on Mon. 14 Oct.

[2] You may assume that so long as $v_s \ll \hbar/m\xi(T)$ where $\xi(T)$ is the GL healing length, the superflow does not contribute appreciably to the free energy of the bulk phase-slip processes.