Physics in Two Dimensions

Motivation:

1) Many specific 2D systems of interest, e.g. cuprates, graphene, helium films, $Sr_2RuO_{4-y}Si$ MOSFETS…, ultracold atomic gases…
2) (almost) unique theatre for realization of TQC.
3) Forces us to rethink just about everything we took for granted in 3D, from scratch

Draft plan of course (note degree of flexibility, at end) omissions (magnetism…)

Textbooks handouts

*Assessment

Meaning of 2D:

In a many-body system, many characteristic lengths:

- Atomic dimension $\left(\sim 3\text{Å}\right)$ or, in e^- gas $k_F^{-1}\left(\sim 1\text{Å}\right)$ (but may be considerably larger e.g. in GaAs heterostructures)
- Thermal *d*B wavelength, $\lambda_{DB} \sim h/\left(mk_BT\right)^{1/2}$ (or $\sim hc/k_BT$ for phonons)
- Elastic mfp
- Inelastic mfp (can be $\sim \mu$)
- correlation length

A given system may be "2D" in one context and "3D" in another. (example: a metal film of thickness $\left(d \sim 100\text{Å}\right)$ at 1 K is thoroughly 2D from the point of view of the phonons $\left(\lambda_{DB} \sim 600\text{Å}\right)$ but still 3D from the point of view of the electrons.)

(note on correlation length)

A further complication: in many cases of interest, we have a 3D matrix containing many 2D planes (e.g. organics, cuprates…). Then an important quantity is the degree of coupling between the planes, both due to tunneling and due to (Coulomb) interactions. (Note that even when a system is entirely 2D in all other respects (e.g. graphene), the C.1. propagates in the (3D) vacuum and hence its (unscreened) 3D form is always prop. $\left|\boldsymbol{r}-\boldsymbol{r}'\right|^{-1}$)

Generally speaking, it is a necessary (and often sufficient) condition to regard a given experimental system as "effectively 2D" if at temperature *T* that

(a) $k_BT \ll$ en. of all nontrivial excitations ⊥'r plane (i.e. $d \ll \lambda_{DB}$)

(b) (in case of multilayers) $k_B T \gg$ interplane coupling.

↑: "interplane coupling" ambiguous! E.g. in case of cuprates, usually $k_B T_c \gg t_\perp$ (single-particle. interplane hopping matrix element), ⇒ in formulating microscopic theory, can take to be "2D": on the other hand, for a macroscopic system, $k_B T \ll$ **total** interplane coupling ⇒ for calculating degree of LRO, etc., must take as 3D (more in lecture 15). $\left[\text{also } k_B T \gg \frac{e^2}{\in r}\right]$

(2D magnetic systems)

Some peculiarities of 2D (vis-à-vis 3D)

(1) Bending energies in d dimensions:

(a) consider KE in single-particle QM: If we try to contain the (normalized) Schrödinger wave function over a distance L, then the KE is $\sim \left(\overline{\nabla \psi}\right)^2 L^d \sim L^{d-2} \left|\psi\right|^2$ which since $L^d \left|\psi\right|^2 \sim 1$ is $\sim L^{-2}$. On the other hand, if the (attractive) potential is restricted to a region $\ll L$ then $\langle V \rangle \propto L^{-d}$. Hence in 1D it is always advantageous to let $L \to \infty$, whereas in 3D it is not: 2D is "marginal" so it is not immediately clear what will happen. (See lecture 3.)

(b) Similarly, if one has an order parameter with continuous symmetry which must be "healed", e.g. from a prescribed value (e.g. 0) at the origin to its bulk value at ∞, the bending energy is now$\propto \left(\overline{\nabla \psi}\right)^2 L^d \sim L^{d-2}$ (note the OP is not normalized!) On the other hand, any bulk energy associated with breaking of the continuous symmetry (e.g. in a Bloch wall in a ferromagnetic insulator, the crystalline anisotropy) $\propto L^d$. Hence in 3D advantageous to shrink the domain wall to a point, whereas in 1D it has finite width. Again, 2D is marginal…

(2) (a somewhat related point): In the theory of 2nd order phase transitions, one would like to introduce a "symmetry-breaking field" $\mathcal{H}$, and then take the limit $V \to \infty$ then $\mathcal{H} \to 0$. What exactly does this mean? We want to take the limit in such a way that the energy associated with the SB field, $M\mathcal{H}$ where $M \equiv N$ is the total magnetization,, is $\gg k_{BT}$ but the **single-spin** energy $\mu\mathcal{H} \ll \in_0$, the minimum collective excitation energy of the system (if this is not so, the extended field is in some sense no longer a "small" perturbation). Since for a continuously broken symmetry $\in_0$ typically $\propto k_{\min}^2 \propto L^{-2}$, while $N \propto L^d$, these requirements imply

$$o\left(L^{-2}\right) \gg \mathcal{H} \gg o\left(L^{-d}\right)$$

This condition is automatically satisfied in 3D by taking (say) $\mathcal{H} \propto L^{-5/2}$, $L \to \infty$. It clearly cannot be satisfied in the same way in 2D (or $d < 2$). This point is related to theorems about the absence of LRO in $d \leq 2$, see lecture 4. More generally, fluctuations tend to be much more important in 2D than in 3D.

(3) Random walks: we will explain this topic in detail in lecture 4, but note for now that in 2D (or of course in 1D) a RW returns infinitely many times to the origin, whereas in 3D the probability of return $\to 0$ as $t \to \infty$. This has profound implications for the theory of localization.

(4) The Coulomb interaction: as noted above, the (virtual) EM field is always 3D, so in real space the CI $\propto r^{-1}$ irrespective of d. However, the F.T. is $\propto q^{1-d}$ (for $d > 1$). As a result, the energy of a charge oscillation (plasmon), which is $\propto \left(q^2 V(q)\right)^{1/2}$, $\to$ const. for $q \to 0$ in 3D but in 2D $\propto q^{1/2}$ and thus $\to 0$. A related point is that there is no analogy of the (uniform) dielectric constant $\in$ for a 2D system: in fact, the screening $\to 0$ as $q \to 0$ (see lecture 15).

(5) Scattering processes for degenerate fermions: this is quite a tricky point. Consider a pair of fermions scattering at T=0 in the presence of the Fermi sea. Suppose they have (small) total momentum $\boldsymbol{P}$ relative to the Fermi sea and (small) total energy E relative to $2E_F$. In both 2 and 3D, conservation of energy means that the available phase space (outside the Fermi sea) $\propto E$; where combined with the DOS for initially exciting the 2nd particle, this gives the standard result that the total scattering probability of a **single** electron of energy $\in$ relative to the Fermion surface $\propto \in^2$. What about conservation of **momentum**? In 3D this does not introduce any other critical factor, because there is a "cone" of momentum-conserving final states available, obtained by rotating the initial states around the total momentum vector. In 2D things are quite different: as $\boldsymbol{P} \to 0$ the scattering becomes restricted to the forward direction.

(6) The "strong-coupling" condition: in many MB systems we find that ∃ a correlation ("healing") length ξ which is proportional to $n^{-1/2}$ where n is the (d-dimensional) density. Generally speaking, the o–of–m criterion for some kind of mean field approach to be valid is that the number of particles in a vol. whose side is ξ is $\gg 1$, i.e. $n\,\xi^d \gg 1$. Thus for 3D the condition is $c.n^{-1/2} \gg 1$, i.e. the **low**-density limit, while in 1D it is $c'n^{1/2} \gg 1$, i.e. the **high**-density limit.. For d=2 the criterion to be in the mean-field limit is independent of density at least to logarithmic accuracy.

(7) However, perhaps the most important difference of all between 2D and 3D (or 1D) lies in the effects of **topology**: this is really what makes 2D unique.

Imagine a particle morning in a d-dimensional space, with the origin excluded. In 1D the situation is trivial; if the particle starts on the R of the origin it stays there forever, and vice versa. In $d \geq 3$, imagine an arbitrary closed trajectory which avoids the origin; such a trajectory can explain arbitrary regions of the space before returning to its starting point. However, any such trajectory can be continuously constricted to a point, and so any transformation induced by it must be "homotopically equivalent" to the identity. (Thus, for example, the orbital Schrödinger wave function must be single-valued.)

An important application is to the interchange of two identical particles: We know that this cannot affect the probable density, so we immediately know that $\psi\left(\boldsymbol{r}_1;\boldsymbol{r}_2\right) = \exp i\alpha\,\psi\left(\boldsymbol{r}_2;\boldsymbol{r}_1\right)$, α real. However, **two** interchanges with the same "sense" are equivalent (up to an irrelevant translation of the COM) to taking one particle completely around the other, and by the above argument, in $d \geq 3$ dimensions the result of this must be the identity. Hence it immediately follows

that in 3 (or more) dimensions $\alpha=0$ or π (mod. 2π), giving the standard Bose or Fermi statistics. (The association of these with integral and half-integral spin respectively requires further (nontrivial!) argument).

In 2D the above argument **does not work**, because it is impossible (while continuing to exclude the origin) to shrink the loop trajectory to a point. Hence in principle (if we know nothing about the nature of the "particles" in question) the "exchange phase" α can be any real number. Of course, if the "particles" are real electrons or atoms we know that they "really" live in 3D space and hence must obey B/F statistics even when physically confined (e.g. by low T) to a plane; but there may be more exotic composite objects which can only be defined within the plane, and these can have a priori, any value of α. Such objects were originally envisaged by Leinaas & Myrheim in 1977, and were subsequently christened "anyons" by Wilczek.

Generally speaking, we will find that the nontrivial effects of topology in 2D are strongly enhanced, in the case of charged particles such as electrons, by a magnetic field $\perp$'r to the plane of motion. The reason is that, in any d, in a nonzero magnetic field (and zero $\boldsymbol{E}$), and in the absence of complicating band-structure effects*, the projection of the semiclassical orbit of an electron (etc.) on a plane $\perp$'r to the field is always closed. Since in the 2D case this is the only plane available for the motion, the orbit itself is automatically closed, which is just the state of affairs which is liable to activate "anyonic" effects.

[Advise to brush up elementary chemistry, particularly of C compounds]

* Which can produce "open" orbits.