Statics and dynamics of the BKT transition

A. Statics

Following KT, we start by considering the statistical mechanics of a single vortex-antivortex pair, neglecting for a moment the effect of all the other pairs. Consider the probability of finding the vortex and antivortex a distance r apart. This is the Gibbs factor $\exp{-\beta U(r)}$ times the phase-space density $2\pi r dr$, and so the mean-square radius $\langle r^2 \rangle$ is in this approximation

$$\langle r^2 \rangle = \int_{r_0}^{\infty} 2\pi r^3 \exp{-\beta(2\mu + 2q^2 \ln(r/r_0)dr} / \int_{r_0}^{\infty} 2\pi r \exp{-\beta(2\mu + 2q^2 \ln r/r_0)dr} \quad (1)$$

$$= \int_{r_0}^{\infty} r^{3-2\beta q^2} dr / \int_{r_0}^{\infty} r^{1-2\beta q^2} dr$$

This expression is divergent if $\beta q^2 < 2$; otherwise, it is given by

$$<r^2> = r_0^2 \left(\frac{\beta q^2 - 1}{\beta q^2 - 2} \right) \quad (2)$$

We see once more that the point $\beta q^2 = 2$, i.e. $k_B T_0 = \pi \rho_s(T_0)(\hbar/m)^2/2$ corresponds to some kind of transition, at least in the naive "single-pair" picture used so far.

What this picture neglects is the screening of "large" vortex-antivortex pairs by other pairs that lie within their separation. We first calculate the average separation d between one pair and the next, or, better, the number per unit area of such pairs, d^{-2}. As we have seen, the thermal probability of finding a pair of radius r is just $\exp{-\beta U(r)}$ and the probability of finding a pair at all therefore $2\pi \int_{r_0}^{\infty} r \exp{-\beta U(r)dr}$: to find the density this must be multiplied by a factor with dimension L^{-4}, which is the square of the inverse "phase space" occupied by a vortex core. Fortunately the exact value of this quantity does not matter very much in the subsequent argument, and we may estimate it as of order r_0^{-4}. Thus

$$d^{-2} \approx 2\pi r_0^{-4} \int_{r_0}^{\infty} r \exp{-2\beta(\mu + q^2 \ln(r/r_0)dr} = \frac{(\pi/r_0^2) \cdot \exp{-2\beta\mu}}{\beta q^2 - 1} \quad (3)$$

The ratio $\langle (r/d)^2 \rangle$ is therefore given by

$$\langle (r/d)^2 \rangle \sim \frac{\pi e^{-2\beta\mu}}{\beta q^2 - 2} \quad (4)$$

At this point it is useful to make some rough estimate of the chemical potential μ, which we recall is of the order of that part of the energy of the vortex that is not associated with flow at large distances but with the formation of the core. We see on dimensional grounds that this must be of the form $A_c \rho_s(\hbar/m)^2$, and we can try to estimate A_c by making some simple variational ansatz for the structure of the core, e.g. $|\Psi(r)| = (r/r_0)\Psi_0$ for $r < r_0$.

This gives $A_c = \pi$, so setting $\beta q^2 \sim 2$ we find $\beta \mu \approx 2$; thus the quantity $\pi \exp -2\beta\mu$ is $\sim \pi e^{-4} \sim 0.06$. In their original paper KT assume that $e^{-2\beta\mu}$ is very small compared to unity: in that case the mean radius of a vortex-antivortex pair is small compared to the inter-pair distance except very close to the temperature T_0 at which the former diverges.

If we take the "Coulomb gas" analog seriously, we should expect that when $r \gtrsim d$, the interaction between the "charges" $\pm q$ would be screened by the polarizability of any other pairs that lie between them. We can anticipate that as a result, the effective superfluid density $\rho_s(T) \equiv q_{\text{eff}}^2 (m/\hbar)^2$ relevant to the vortex-antivortex interactions in the limit $r \to \infty$ will be normalized downwards from its MF value (which I will denote as above by $\rho_s^0(T)$). Since near T_0 it is the very long-distance behavior of U(r) that determines $\langle r^2 \rangle$, we would expect that the formula for the latter would be as above but with $q \to q_{\text{eff}}$, i.e. $\rho_s^0(T) \to \rho_s(T)$. Thus, we expect to find the "vortex-unbinding" transition occurring at the point

$$k_B T_0 = (\pi/2)\rho_s(T_0)(\hbar/m)^2 \tag{5}$$

where however $\rho_s(T_0)$ is the *normalized* superfluid density, i.e. that appropriate to $r \gg d$. Since it is usually this quantity that is measured in actual experiments (cf. next lecture) and since, as we will see, on the high temperature side of the transition there is no superfluidity, we can make the remarkable prediction that if the transition occurs at a temperature T_0, the *jump* $\Delta\rho_s$ in the experimentally measured superfluid density per unit area is given by

$$\boxed{\Delta\rho_s = \frac{2}{\pi}\left(\frac{m}{\hbar}\right)^2 k_B T_0} \tag{6}$$

The core of the KT paper (section 2) is devoted to filling in the details of the argument leading to this result, which I now sketch.

Consider a vortex-antivortex pair with separtion r (in an arbitrary direction), and imagine applying to it a field $\mathcal{E}$ in (say) the x-direction. The polarizibility is defined as $\langle qr\cos\theta \rangle / \mathcal{E}$ where θ is the angle with the x-axis. The probability of a given angle θ is now weighted with a factor $\exp -\beta \mathcal{E} qr\cos\theta$, so the expectation value is

$$\langle qr\cos\theta \rangle = \int (qr\cos\theta)e^{-\beta\epsilon qr\cos\theta}d\theta / \int e^{-\beta\epsilon qr\cos\theta}d\theta \tag{7}$$

which in the limit of small ϵ is $\beta q^2 r^2 \epsilon \int \cos^2\theta \, d\theta / = 2\pi = \frac{1}{2}\beta q^2 r^2 \epsilon$. Thus the polarizability of the pair[1] is just $\frac{1}{2}\beta q^2 r^2$.

To find the polarizability $\chi(r)$ of the system we must multiply the quantity just calculated by the number of pairs per unit volume with separation in the range $(r, r + dr)$. Here we have a problem of self-consistency: the definition of the dielectric constant which we will give below[2] $\left(\equiv 1 + 4\pi\chi(r) \right)$ is actually the factor by which the *force* between the

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[1]One might wonder whether the application of the field would affect the probability as a function of r. However, this effect is of higher order in $\mathcal{E}$ and thus negligible.

[2]I use CGS units (in which $\epsilon = 1 + 4\pi\chi$) to be consistent with the previous notation for the "charges."

two charges (in the 2D Coulomb gas $2q^2/r$) is divided, hence the effective *potential* is not $2q^2/\epsilon(r)$ but rather $2q^2 \int_{r_0}^r dr'/r'\epsilon(r')$. The number of pairs per unit area with separation between r and $r + \delta r$ is then

$$dn(r) = \frac{2\pi r\delta r}{r_0^4} \exp -\beta U_{\text{eff}}(r) \tag{8}$$

$$= \frac{2\pi r\delta r}{r_0^4} \exp -2\beta \left(\mu + q^2 \int_{r_0}^r \frac{dr'}{r'\epsilon(r')} \right)$$

and the change in the dielectric constant due to these pairs (which I write as $\delta\epsilon[\delta r]$ rather than $d\epsilon(r)$ for a reason that will become clear) is

$$\delta\epsilon[\delta r] = 4\pi^2 \beta q^2 r^3 \frac{\delta r}{r_0^4} \exp -2\beta \left(\mu + q^2 \int_{r_0}^r \frac{dr'}{r'\epsilon(r)} \right) \tag{9}$$

We now come to a slightly delicate point, which (as far as I can see) is not discussed in the original KT paper[3]. The quantity $\delta\epsilon[\delta r]$, which we have just evaluated, is actually the contribution to the $(r = \infty)$ dielectric constant *from pairs with separation in the range $(r, r + \delta r)$*. On the other hand, the "$\epsilon(r)$" that stands in the exponent on the RHS of the equation is the factor by which the interaction of pairs *at* separation r is reduced. Is it legitimate to identify $\delta\epsilon/\delta r$ with the simple derivative $d\epsilon/dr$ as KT do? I think the answer is that this is not obvious, but that correcting it (say by taking the ratio of the two quantities to be constant of order 1) would affect only some of the intermediate definitions, not the overall structure of the results. (In any case, as we shall see, the results obtained by this approximation can also be obtained by an alternative method).

With this approximation, then we obtain for $\epsilon(r)$ a single integrodifferential equation of the form

$$\frac{d\epsilon}{dr} = 4\pi^2 \beta q^2 \frac{r^3}{r_0^4} \exp -2\beta \left(2\mu + q^2 \int_{r_0}^r \frac{dr'}{r'\epsilon(r')} \right) \tag{10}$$

with the boundary condition $\epsilon(r_0) = 1$ (since at scale r_0 there is no screening). In their original paper KT solve this equation by approximating the integral in the exponent on the RHS by $q^2(\ln r)/\epsilon(r)$. However, as shown by Young[4] a few years later, it is not necessary to make this approximation. In fact, let us introduce the variables

$$x(r) \equiv \left[\beta q^2/\epsilon(r) \right] - 2 \tag{11}$$

$$y(r) \equiv 4\pi \exp \left[-\beta \left(\mu + q^2 \int_{r_0}^r \frac{dr'}{r'\epsilon(r')} \right) - 2\ln(r/r_0) \right]$$

Then eqn. (10) is equivalent to the pair of equations

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[3]It is briefly discussed in Young's paper (see below).

[4]J. Phys. C **11** L453 (1978).

$$\frac{dy}{d(\ln\ r)} = -xy \tag{12}$$

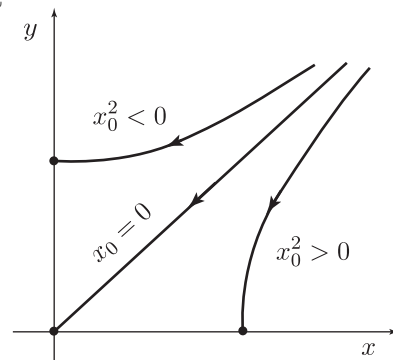$$\frac{dx}{d(\ln\ r)} = -\frac{(x+2)^2}{4}y^2 \tag{13}$$

with the boundary conditions $x(r_0) = \beta q^2 - 2$, $y(r_0) = 4\pi e^{-\beta\mu}$. We will be interested (cf. below) in the limit of $x(\infty) = 0$, and since y is "small" this means x is small everywhere; thus we can legitimately replace[5] the factor $(x+2)^2/4$ by 1, and the equations reduce to $(\tau \equiv \ln r)$

$$\frac{dx}{d\tau} = -y^2, \qquad\qquad \frac{dy}{d\tau} = -xy \tag{14}$$

These equations (and actually the more exact equations above) were originally derived by Kosterlitz[6] from a standard RG treatment of the Coulomb-gas problem. It is easy to check that they are solved by the $x - y$ relation

$$x^2 - y^2 = \text{const.} \equiv x_0^2 \tag{15}$$

(where however x_0^2 may have either sign). We expect that
the parameter $x_0^2 \left[\equiv x^2(r_0) - y^2(r_0)\right] \equiv \beta q^2 - 2 - 4\pi e^{-\beta\mu}$
will be a smooth function of temperature, decreasing
with increasing temperature. This equation is familiar
as a solution of the RG equations for various types of
phase transition: for $x_0^2 > 0$ it represents a (half-) hy-
perbole oriented along the x-axis and for $x_0^2 < 0$ one
oriented along the y-axis, with the special value $x_0 = 0$
corresponding to a straight line of $45°$ slope through
the origin. The "flow" with $\tau \equiv \ln r$ is as indicated,
so that the end-points represent the values of x and y
for $r \to \infty$. Thus, the value of $x(\infty) \equiv \beta q^2/\epsilon(\infty) - 2$
at the "separatrix" $x_0 = 0$ is 0, and remembering that for the superfluid instantiation
$\beta q^2/\epsilon(\infty)| \equiv \pi\rho_s(T)(\hbar/m)^2$ where $\rho_s(T)$ is the *experimentally observed* superfluid density,
we find that at the temperature T_{KT}, defined as that value of T for which x_0 passes through
zero, we have

$$\frac{\pi}{2}\rho_s(T_{\text{KT}}^{(-)})\left(\frac{\hbar}{m}\right)^2 = k_B T_{\text{KT}} \tag{16}$$

which is (one part of) the result we obtained more intuitively above.

One might, however, legitimately ask whether it is obvious that the experimentally
observed superfluid density is zero above T_{KT} (the other "half" of our earlier statement).
One way of seeing that it is (which anticipates some considerations to be developed further
in the next lecture) is to note that we already saw in lecture 11 that 3D superfluidity can

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[5]Actually the equations have an exact solution as they stand; see AHNS appendix D.
[6]J. Phys. C **7**, 1046 (1974)

be destroyed by the nucleation and growth of vortex rings. The analogous process in 2D is that vortex-antivortex pairs will be pushed apart by the superflow and, again, this will lead to a change in the winding number and thus the destruction of superflow. The force created by a superflow $\mathbf{v}_s$ on a vortex-antivortex pair is independent of r and is just the sum of the two Magnus forces, i.e. $2\rho_s v_s(\hbar/m)$. Now the local "spring constant" $\partial^2 U_{\text{eff}}/\partial r^2$ is $2q^2/r^2\epsilon(r)$, and the "radial polarizability" (the average change in the separation induced by the Magnus force) is the inverse of this and thus at large distances is simply proportional[7] to r^2. Thus, the average "polarizability" is finite or infinite depending on whether $\langle r^2 \rangle$ is finite or not. But we have already seen that the condition that it is finite is $\beta q^2/\epsilon(\infty) > 2$, i.e. that $T < T_{\text{KT}}$. For $T > T_{\text{KT}}$ any vanishingly small superflow will push the pairs apart and thereby lead to its own demise.

An obvious extension of this argument suggests that even below T_{KT} there will be a maximum "external" superfluid velocity beyond which superflow can push "most" vortices apart, i.e. an effective critical velocity. We will investigate this below, but note that this question makes it interesting to investigate whether and how the quantity $\langle r^2 \rangle$ approaches ∞ as $T \to T_{KT}$ from below. Close to the transition $\langle r^2 \rangle$ should be inversely proportional to $\beta q^2/\epsilon(\infty) - 2 \equiv x(\infty)$, so the question reduces to how $x(\infty)$ approaches zero for $x_0 \to 0$. (Recall that in the naive "noninteracting" approximation (in which of course $\epsilon(\infty) \equiv 1$) $x(\infty)$ is simply linear in $T_c - T$). Actually, provided that we are prepared to accept that the point $y \to 0$ corresponds to $r \to \infty$ (which is clear from the definition of y) then we can read off immediately the result that $x(\infty) = x_0$, i.e. $x(\infty)$ goes continuously to zero as $T \to T_{KT}$ from below, just as in the noninteracting approximation (but differently from its behavior in the approximate theory of the original KT paper, cf. their eqn. (27)). Hence $\langle r^2 \rangle$ indeed approaches ∞ as the KT transition is approached from below.

Returning to the Kosterlitz equations (14) and inspecting equation (15) and the figure, we see that even without an explicit solution for x, y as functions of τ we can draw, essentially on dimensional grounds, the conclusion that the value τ_0 of τ at which the deviation of the flow from the 45° line in the figure becomes significant is proportional to x_0^{-1}, where x_0, from the arguments above is proportional to $|(T_{\text{KT}} - T)|^{1/2}$. Thus, the characteristic *distance* ξ at which the screening starts to matter qualitatively has the temperature-dependence (on both sides of the transition)

$$\xi \sim \exp b/|T_{\text{KT}} - T|^{1/2} \tag{17}$$

This dependence is very different from the behavior $\xi(T) \sim |T_c - T|^{-\nu}$ of the correlation length associated with a standard second-order phase transition.

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[7]since for large r $\epsilon(r)$ is nearly constant.

B. Dynamics

It is rather difficult to test the KT theory directly in static experiments on helium films[8], as the most interesting predictions relate to the (macroscopic) superfluid density $\rho_s(T)$, and there are obvious difficulties associated with measuring this quantity for a thin film. In fact, it is much easier to measure ρ_s by a finite-frequency experiment, either of the Andronikashvili type (see below) or third sound; the principal difference between these is that in the former conditions are homogeneous in space, and in particular we expect the supercurrent to be divergence-free, while third sound corresponds not only to a finite divergence of the supercurrent but to a spatially varying film thickness. Whichever we use, a crucial point is that in view of the divergence of length scales (and hence presumably of time scales) at the KT transition we cannot assume a priori that the quantity we measure in a finite frequency experiment will necessarily be the static superfluid density. Consequently, to make meaningful contact with experiments we need to extend the KT theory to cover dynamical phenomena. This was done in a number of papers in the late 70's; the most complete treatment is that of Ambegaokar et al.[9], and I shall follow that here (but will confine myself to the spatially homogeneous situation realized in Andronikashvili-type experiments).

Actually it turns out there are two major regimes of interest, in which the physical mechanism of response to an external a.c. probe are rather different: (a) In the regime of large amplitudes and low frequencies below T_c (only), the superfluid density as measured e.g. in an Andronikashvili experiment (cf. below) is to a good approximation the static value, and the major mechanism of dissipation is the splitting of vortex pairs by the superflow (the 2D analog of the LF mechanism briefly discussed in lecture 10). (b) In the regime of low amplitudes and high frequencies, either below or above T_c, the measured "superfluid density" is in general *not* the static value, and the major mechanism of dissipation is the motion of bound and (above T_c) free vortices. In regime (a), provided we are interested in the dominant terms in the dissipation (in particular in their dependence on v_s) and not in the detailed numbers, we can actually make predictions on the basis of the static theory; so I will investigate this regime first.

The analysis of the dissociation of vortex pairs by an external superflow is straightforward: In the presence of a background flow v_s, the energy of the pair acquires an extra term due to the Magnus force, so the total energy is

$$U_{\text{eff}}(r) = 2 \left(\mu + q^2 \int_{r_0}^{r} \frac{dr'}{r' \epsilon(r')} \right) - \frac{2h}{m} \rho_s(T) v_s r \tag{18}$$

This has a maximum at the value r_c given by

$$q^2 / r \epsilon(r) = (h/m) \rho_s(T) v_s \tag{19}$$

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[8] As we shall see in lecture 12, a rather direct test is now available using ultracold Bose gas condensates.
[9] Phys. Rev. B **21**, 1806 (1980). (hereafter ANHS)

or equivalently, since $q^2 \equiv \pi \rho_s^0(T)(\hbar/m)^2$ and $\rho_s/\rho_s^0 \equiv \epsilon^{-1}(\infty)$,

$$r_c = \frac{\hbar}{mv_s} \frac{1}{\epsilon(r_c)} \sim \hbar/mv_s \tag{20}$$

(since $\epsilon(r)$ does not change much between r_c and ∞). Now the thermal probability of finding the r-value of a given vortex pair in the range $(r, r + dr)$ is proportional to $r^{-\eta}$, where η is close to 4 near the KT transition, and the DOS factor is $r\,dr$, so at first sight one would think that the Arrhenius-Gibbs factor (i.e. the probability of finding the system "near" the saddle-point) should be proportional to r_0^{-3}. This is indeed true.

However, it turns out (as in many such problems) that the relevant "attempt frequency" ν_0 is proportional to $\sqrt{(\partial^2 U/\partial r^2)}_{r=r_c}$, and this quantity is clearly $\sim r_c^{-1}$. Hence at the end of the day it turns out[10] that the rate of nucleation of free vortices from bound vortex-antivortex pairs is actually proportional to r_c^{-4}, i.e. to v_s^4. This, however, is not the end of the story, since one has to also take into account the recombination process of the free vortices so produced. Since this rate is proportional to n_f^2, in a steady state $n_f \propto v_s^2$. Moreover, it turns out that the recombination contributes to the decrease dv_s/dt of v_s an amount proportional to $n_f v_s$. As a result of these considerations Ambegaokar et al. conclude that the actual rate of decrease of the supercurrent near T_{KT} scales as v_s^3 (or in the case where $\mathbf{v}_n \neq 0$, $(v_s - v_n)^3$) (with corrections proportional to $x_0(T)$ in the exponent $\delta = 3 + \frac{1}{2}x_0(T)$).

We now turn to regime (b), that is the regime of small amplitudes and appreciable frequencies. In this regime nucleation processes are negligible and both dissipation and reactive effects come from the small-amplitude response of bound and, above T_c, free vortices to the oscillating "field". To evaluate this we need to discuss the dynamics of the vortices. Let's consider a single vortex-antivortex situated in a potential field $U(r)$, which may include a "Magnus" contribution. The resulting force has in general four contributions:

(1) The potential gradient $= \boldsymbol{\nabla} U$

(2) The Magnus force $(nh/m)\rho_s(T)\big(\hat{z} \times (\mathbf{v} - \mathbf{v}_s)\big)$ (where $n \equiv \pm 1$ labels the vortex chirality)

(3) A dissipative force due to the normal component:

$$\boldsymbol{F}_d = -B(\mathbf{v} - \mathbf{v}_n)$$

(4) A nondissipative force due to the normal component:

$$\boldsymbol{F}_{nd} = -B'\, n\hat{z} \times (\mathbf{v} - \mathbf{v}_n)$$

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[10]The much more sophisticated calculation of Ambegaokar et al. reproduces this result to within logarithmic factors, see their eqn. (4.10c).

Note that $\mathbf{v}_n$ can be defined, e.g. by the rotation of the substrate, even in the limit that the normal density $\rho_n \to 0$.

(In the above, I adopt the convention that $\mathbf{v}_s$ is the *uniform* background superflow, and any contribution to the Magnus force seen by a particular vortex from the field of its neighbor in a pair is included in $U(r)$.)

A fundamental assumption of the ANHS theory is that the motion of the vortices is overdamped, i.e. that any "inertial" term (proportional to dv/dt) can be neglected.[11] In that case the motion of a given vortex can be obtained by setting the total force, i.e. the sum of (1)-(4), equal to zero. Let's do this, first, for a single isolated vortex ($U(r) = 0$). From symmetry the solution must have the general form

$$\mathbf{v} = \mathbf{v}_s + C(\mathbf{v}_n - \mathbf{v}_s) + (D/kT)\left(\rho_s^0(k_B/m)(n\hat{z}) \times (\mathbf{v}_n - \mathbf{v}_s)\right) \qquad (21)$$

where the coefficients C and D can be obtained in terms of B and B$'$ (see ANHS equations (2.45)); an explicit factor of $k_B T$ is introduced into D for subsequent convenience. The important point to note is that the first two terms on the RHS of this equation are independent of the sign of $\hat{\mathbf{n}}$ and therefore move a vortex and its partner antivortex in the same way. It is clear that such a motion cannot change the value of the supercurrent, and these terms can therefore be neglected for our purposes. Doing this, we can write

$$\boldsymbol{v} = \mu \boldsymbol{F}_M \qquad (22)$$

where $\mu \equiv D/k_B T$ is a "mobility"[12] and $\boldsymbol{F}_M \equiv n\rho_s^0(h/m)\hat{\boldsymbol{z}} \times (\mathbf{v}_n - \mathbf{v}_s)$ is an "effective" Magnus force. At this point we could, if we wished, introduce also noise terms (cf. AHNS, eqn (2.7)), but this is actually not necessary for our purposes.

Now consider a vortex pair, with a relative separation $\mathbf{r}$ which may depend on time. The motion of $\mathbf{r}$ will "see" twice any effective Magnus force, plus a potential term which is twice $\boldsymbol{\nabla}U$, and it is highly plausible (and can be proved by a more detailed consideration, see AHNS) that it will respond in the same way to both, that is,

$$\dot{\mathbf{r}} = 2\mu(-\boldsymbol{\nabla}U + \boldsymbol{F}_M) \qquad (23)$$

In particular, if $\mathbf{v}_n$ and hence the Magnus force oscillate, e.g. due to the motion of a substrate to which $\mathbf{v}_n$ is tied, we would expect to excite (overdamped, cf. below) harmonic motions of $\mathbf{r}(t)$, and thereby dissipate energy.

The final ingredient needed in the theory is that if we have a number of pairs present and they are changing their $\mathbf{r}$-values, the effect is to change the average value $u_s(t)$ of the total superfluid velocity using eqns. $(43 - 4)$ of lecture 10 we find:

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[11]If this is not the case one has to decide on the correct expression for the "effective mass" of the vortex, a question on which there is still considerable debate.

[12]Do not confuse with the vortex chemical potential, which will not appear below.

$$\frac{d\boldsymbol{u}_s}{dt} = -\frac{1}{A}\sum_i (2h/m)\hat{\boldsymbol{z}} \times \dot{\mathbf{r}}_i \sim K(\mathbf{v}_n - \boldsymbol{u}_s) \tag{24}$$

(where the constant K may be complex, cf. below).

To obtain the essentials of the results we now proceed in a somewhat "hand-waving" way. Consider a pair that initially has spacing r_0 (do not confuse with the core radius!). In the initial state, we can conceive the term in $\boldsymbol{\nabla}U$ as being balanced by thermal effects. Thus, the system is effectively subject to a harmonic potential of the form

$$U_{\mathrm{eff}}(r) \sim \frac{1}{2}\frac{\partial^2 U}{\partial r^2}(r - r_0)^2 \tag{25}$$

where the "spring constant" $k \equiv \partial^2 U/\partial r^2 = Kr_0^{-2}$ where K is approximately $2q^2 \cong 4\beta^{-1}$. Ignoring directional factors, etc., we therefore find for the variable $x(t) \equiv r(t) - r_0$ the equation

$$\dot{x} = \mu(F_m - 4\beta^{-1} r_0^{-2} x) \tag{26}$$

For a sinusoidal driving force with frequency ω this gives for the oscillation amplitude (recall that $\mu \equiv D/k_B T$)

$$x = \mu F_M/(i\omega - 4Dr_0^{-2}) \tag{27}$$

and since the rate of change du_s/dt of the mean superfluid velocity u_s is proportional to $\dot{x}$ (see (24) above) the contribution to u_s from vortex-antivortex pairs of radius r_0 is similarly proportional to $(i\omega - Dr_0^{-2})^{-1}$. It is intuitively clear that the maximum contribution to the dissipation will occur for $r_0 \sim \sqrt{4D/\omega}$; in fact, pairs with radius much less than this will contribute essentially statically to the polarization, while pairs much larger than this will hardly respond at all. Arguing along these lines (and putting in all the necessary factors!) AHNS conclude that to a good approximation the contribution $\epsilon_b(\omega)$ of the bound pairs to the finite-frequency dielectric constant is given by[13]

$$Re\,\epsilon_b(\omega) = \tilde{\epsilon}\left(r = (cD/\omega)^{1/2}\right) \tag{28}$$

$$Im\,\epsilon_b(\omega) = \frac{1}{4}\pi(r\,d\tilde{\epsilon}/dr)_{r=(cD/\omega)^{1/2}}$$

where $\tilde{\epsilon}(r)$ is the *static* dielectric constant and the constant c, which according to the above "hand-waving" argument should be ~ 4, actually turns out to be close to 14. Note that D is to a good approximation simply a constant (i.e. independent of T and r). As we will see

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[13]The derivation of the formula for $Im\epsilon_b(\omega)$), in particular, requires some nontrivial further work. All that the above "hand-waving" argument really does is to make it plausible that it is pairs with $r \sim \sqrt{D/\omega}$ which play the most important role.

shortly, the real and imaginary part of $\epsilon(\omega)$ can be measured e.g. in an Andronikashvili-type experiment; we see that in the limit $\omega \to 0$ the measured quantity is the static $\tilde{\epsilon}(\infty)$, i.e. the static superfluid density $\rho_s(T)$, as we should expect.

All the above refers to the contribution of the "bound" vortex-antivortex pairs; those will contribute, for any ω, for $T < T_c$, and also for $T > T_c$ provided that the characteristic length $(cD/\omega)^{1/3}$ is less than the length ξ_+ at which the pairs become "effectively" unbound. In addition, above T_c there will be a contribution $\epsilon_f(\omega)$ to the dielectric constant from the *free* vortices. This can be obtained very simply by putting the "restoring force" Dr_0^{-2} in the above equation to zero, so that the vortex coordinate is simply given in F.T. form by

$$x = \mu F_M / i\omega \tag{29}$$

In this way we obtain

$$\epsilon_f(\omega) = (4\pi n_f q_0^2 Dk/k_B T)/i\omega \cong 2\pi n_f \, D/i\omega \tag{30}$$

i.e. a purely dissipative response. Note that here $q_0^2 \equiv \pi(\hbar/m)^2 \rho_s^0(T)$, so it is finite in the region $T_K T < T < T_0$ where T_0 is the 3D transition temperature. n_f in the above formula is the density of free vortices, which is proportional[14] to ξ_+^{-2} and tends to zero very rapidly $(\sim \exp -t^{-1/2})$ as $T \to T_{KT}$ from above. Of course, there will be "background" contributions to $\epsilon(\omega)$ of the same form as (29) from (e.g.) dissipation in the normal component, but they would be expected not to be strong functions of temperature.[15]

If we ignore the "free" contribution, we reach the following qualitative conclusions: For a given oscillation frequency, ω, there exists a characteristic length $r_c \equiv (cD/\omega)^{1/2})$ (where $c \cong 14$ according to the detailed calculations of AHNS); the value of ϵ at this length scale determines the value of the effective superfluid density (Re ϵ) and dissipation (Im ϵ) for the experiment. Now, we have to bear in mind that both below and above T_{KT} there is a characteristic length associated with the screening, ξ_- and ξ_+ respectively; in both cases ξ is proportional to $\exp bt^{-1/2}$, where b is a constant of order 1 and $t \equiv |1 - T/T_c|$, and hence diverges very strongly in the neighborhood of T_{KT}. The physical significance of ξ_- is (cf. above) that it is the length at which $\epsilon(r)$ effectively attains its "macroscopic" value $\epsilon(\infty)$, while ξ_+ is the length scale at which, above T_{KT}, the pairs effectively become unbound. Suppose now that we work at fixed frequency (hence fixed r_c, since we expect $D(T)$ not to be particularly sensitive to T in the region of the transition) and vary T. Then for T well below T_{KT} (and realistic values of the parameters, see below) we expect that $r_c \gg \xi_-$, and thus (a) the real part of ϵ should take its "macroscopic" value $\epsilon(\infty)$ (in the superfluid density measured in the experiment should be the static value), and (b) since ϵ is not varying appreciably with r, the imaginary part of ϵ, i.e. the dissipation, should be very small. As we approach T_{KT}, $\xi_-(T)$ will rise sharply and eventually exceed r_c; above T_{KT},

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[14]$n_f \sim \int_{\xi_+}^{\infty} n(r) d^2\mathbf{r} \sim \int_{\xi_+}^{\infty} r \frac{dr}{r^4} \sim \xi_+^{-2}$

[15]In fitting their experimental results to the data, Bishop and Reppy (see below) take into account the contribution to ϵ_f with one overall adjustable constant (F)

$\xi_+(T)$ decreases rapidly and eventually falls below r_c. In the "critical" region close to T_{KT} where r_c is smaller than ξ_- or ξ_+, we should expect (a) the "effective" value of ϵ rises with T (and in fact $\to \infty$), so that when we come out of this region on the high-T side the superfluid density is zero, and (b) since ϵ is a function of r in this regime, we should get strong dissipation. Actually, by combining the various formulae used above we can derive some specific relations, which conform to the above conditions; these are quoted by Bishop and Reppy (their eqns. (11)–(12)), but I do not reproduce them since it seems there is almost certainly at least one typographic error in their formula.