

The superconductor-insulator transition in dirty metallic films¹

Since around 1980 it has been appreciated that if in certain kinds of metallic film one varies the degree of disorder, thickness, electrostatic doping and/or the magnetic field while keeping the chemical composition constant, the system may make a transition between a superconducting and an insulating state. Interestingly, there are two major approaches to explanation of these phenomena, related respectively to the effects of disorder on the single-electron spectrum discussed in lectures 4-7 and to the fluctuations of the phase of the superconducting order parameter that have been the subject of lectures 8-12 (through as we shall see, the relevant fluctuations in this case are quantum rather than classical in nature).

The films in question are typically obtained by sputtering (or sometimes by vacuum deposition at helium temperatures) of a metal such as In, Mo or Bi, sometimes accompanied by a nonmetallic element such as N, O or Si, on a substrate such as Ge or AlO. Typical examples are Bi, In_xO_y , Mo_xGe_y , Au_xSi_y ...; note that none of these elements form localized magnetic moments. Magnetic fields are typically in the range 0.1-1 T. Thicknesses d are typically in the range 4-80 Å (thus, d may be either small or large compared to the Cooper-pair radius, which we recall in a dirty metal is $\sim (\xi_0 l)^{1/2}$ (ξ_0 = pair radius in pure metal, l = elastic mfp). A very important characteristic of any given film is R_{\square} , the “sheet resistance” (resistance per square) in the (high-temperature) normal state (which is usually nearly temperature-independent): the general order of magnitude of this quantity at the S-I transition is the “quantum unit of resistance”, $R_Q \equiv h/4e^2 \cong 6.45\text{k}\Omega$; we will see below that this may not be an accident.

While all the relevant samples are amorphous (i.e. noncrystalline), an important question is the degree of homogeneity. This may vary considerably between samples of different composition and/or deposited on different substrates; however, it seems unlikely to depend strongly on film thickness (or of course on magnetic field). The general belief is that for e.g. Bi deposited on a Ge substrate the disorder is likely to be on the atomic scale, while other kinds of film may be inhomogeneous on a more mesoscopic scale, say $\sim 10^2$ Å; such samples are often called “granular”, and sometimes modelled as collections of metallic “islands” connected by Josephson junctions. Ideally of course we should try to characterize the degree of homogeneity by some technique such as atomic force microscopy, but this has not always been done, particularly in the earlier experiments, and thus there is often some uncertainty about the degree of granularity.

For any given film the raw data typically consists of the resistance $R(B, T)$ (often expressed as a sheet resistance) as a function of temperature T and magnetic field B . (A few experiments have also measured the Hall resistance.) In the early experiments² of the

¹A readable and up-to-date reference is Lin et al., *Physica C* **514**, 130 (2015).

²Haviland et al., *Phys. Rev. Lett.* **62**, 2180 (1989)

Minnesota group on Bi quenched on to an α -Ge substrate the data are extremely clean, as shown in the figure. Thin films show a strong upturn in the resistance at low temperatures, apparently tending to an insulating state in the limit $T \rightarrow 0$, while thicker ones experience an abrupt decrease of resistance to an unobservably small value at a (thickness-dependent) temperature of a few K, indicating a transition to the superconducting state. Since the dependence of the (inverse) sheet resistance R_{\square} on thickness is presumably monotonic, one can say that films of high R_{\square} become insulating while those of lower R_{\square} (higher conductance) become superconducting. Two very significant features of the data are that (1) no normal-metallic state appear to exist in the limit³ $T \rightarrow 0$, and (2) the “critical” normal-state sheet resistance R_c which marks the separatrix between superconducting and insulating behavior is very close to the quantum unit of resistance R_Q . I postpone for the moment the question of the behavior in magnetic fields (which was not examined in this particular set of experiments).

Not all thin dirty metallic films show quite so clear-cut a behavior as a function of thickness and temperature as the above ones; for example, in some cases the behavior of $R(T)$ is nonmonotonic, and in others it appears to flatten off as a function of T above the lowest temperature reached (typically ~ 0.1 K), indicating the possibility of a normal-metallic state (in zero magnetic field) in the limit $T \rightarrow 0$; this happens not only in some films that are probably granular, but even in some samples of Bi on Ge, and in this case the flattening persisted down to the much lower minimum temperature of the experiment, less than 50 mK. This last data is fairly recent and probably needs further confirmation (cf. Goldman, op.cit.); it is of course not taken into account in the theoretical analyses that were conducted earlier.

Turning now to possible interpretations of the data, let’s start with an old and classic observation of Anderson: *weak* nonmagnetic disorder should not affect the superconducting transition qualitatively. The reason is that while the plane-wave states $\exp i\mathbf{k} \cdot \mathbf{r}$ are no longer eigenstates of the single-particle Hamiltonian, and hence cannot (at least prima facie) be used to form Cooper pairs, it is still possible to find pairs of exact eigenfunctions of the single-particle Hamiltonian $\hat{H}_0 \equiv -\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r})$ that are related to one another by time reversal. For example, if as is normally the case one can choose the eigenfunctions of \hat{H}_0 to be real functions $\varphi_n(r)$ such that $\hat{H}_0\varphi_n(r) = \epsilon_n\varphi_n(r)$, then a suitable choice of pairs would be simply (φ_n, \uparrow) and (φ_n, \downarrow) with \uparrow, \downarrow the spin eigenstates. One then forms the groundstate along the lines of BCS, but replacing the plane waves $\mathbf{k} \uparrow, -\mathbf{k} \downarrow$ by $(\varphi_n, \uparrow), (\varphi_n, \downarrow)$, i.e.

$$\Psi_0 = \prod_n (u_n + v_n a_{n\uparrow}^+ a_{n\downarrow}^+) |\text{vac}\rangle \quad (1)$$

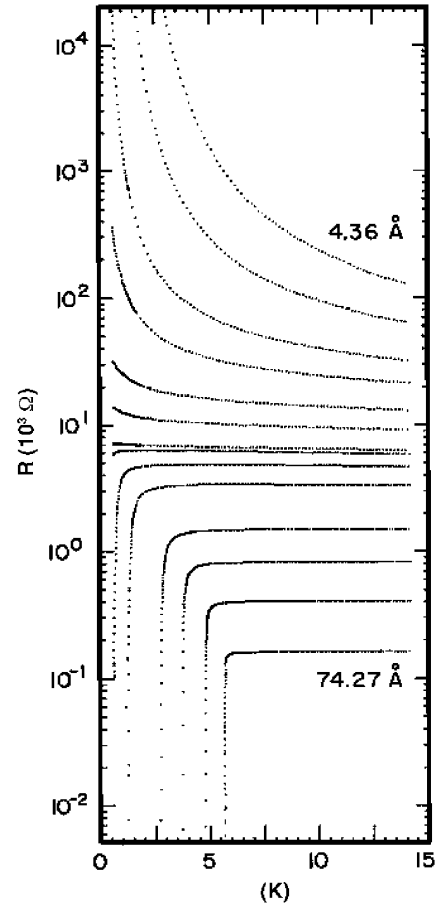
The subsequent calculations proceed exactly as in the original BCS theory; since the average single-particle DOS near the Fermi energy is little affected by the disorder, and the (coarse-grained) matrix elements $V_{nn'}$ for pair scattering do not vary much from the free-space elements $V_{\mathbf{k}\mathbf{k}'}$, it follows that the thermodynamic properties of the disordered

³Assuming that the curves very close to the separatrix turn up or down at a temperature below the lowest accessible in the experiment.

system, including T_c , are very close to those of the original pure one.⁴ (Of course, there may be substantial effects on quantities such as the superfluid density $\rho_s(T)$ which relate to transport.)

However, it is known that even in 3D sufficiently strong disorder will *localize* the single-particle states near the Fermi energy; and we saw in lectures 4-7 that in 2D such localization inevitably occurs in the limit $T \rightarrow 0$ however weak the disorder. So what happens to superconductivity at this point? In the normal phase the system is certainly an insulator; and it is extremely tempting to argue that even if Cooper pairs form from the localized single-electron states, they will be, like these states, unable to propagate, and thus the paired system will remain an insulator. Remarkably, this plausible conclusion turns out to be *wrong*!⁵ The basic reason is that while the single-particle wave functions no longer extend over the whole volume of the sample as in the pure metal, and hence in (e.g.) a ring geometry cannot be affected by an AB flux, the two-particle (Cooper pair) wave function, regarded as a function of the COM variable, does so extend and thus *is* sensitive to an AB flux. So we get the remarkable prediction that in a 3D system in which in the absence of an attractive interaction the disorder is sufficient to localize all the single-particle states, switching on an attractive interaction (e.g. due to the exchange of virtual phonons) will produce the following result: at temperatures $>$ some T_0 , where T_0 is generally speaking of the order of the T_c of the pure (non-disordered) system, the system will be a good insulator; however, the moment that T falls below T_0 it will become not just conducting but superconducting! Whether this state of affairs has been seen in existing experiments is debatable;⁶ in any case it may be possible in future to verify (or refute) the prediction using ultracold Fermi atoms in a disordered optical lattice.

In any case, one might guess that when the localization of the single-particle states gets too strong, it might be energetically disadvantageous to form Cooper pairs at all. A very rough criterion might be that the energy spacing of the single-particle levels in a volume of the order of the localization length ξ_{loc} becomes comparable to the bulk energy gap Δ_0 ;



⁴If anything, the “smoothing” of the matrix elements by disorder tends to *increase* T_c .

⁵Bulaevskii and Sadovskii, JETP Lett. **39**, 11 (1984); Ma and Lee, Phys. Rev. B **32**, 5658 (1985)

⁶See Sadovskii, Phys. Rep. **282**, 225 (1997).

in 2D this would then give us the condition for the destruction of superconductivity

$$\xi_{\text{loc}}^2 \lesssim \frac{\hbar^2}{2m\Delta_0} \sim k_F^{-1}\xi_0 \quad (2)$$

where ξ_0 is the Cooper-pair radius in the *pure* material. Since in a 2D system the localization length is exponentially large ($\mathcal{O}(l \exp(k_F l))$) even at $T \rightarrow 0$, it seems unlikely that this transition would be experimentally accessible, except perhaps in the very dirtiest materials.

Unfortunately, as always in the localization problem, it turns out that this consideration is anyway too simple: one needs to take into account the interplay of disorder and interaction effects which, as discussed in lecture 7, changes the low-energy single-particle energy spectrum qualitatively. The ensuing calculations, for which field-theoretic methods are unavoidable, are discussed in detail by Finkelstein.⁷ Without going into the details, we note (a) that the primary physical effect responsible for the suppression and eventual destruction of superconductivity in this approach turns out to be that since a pair of electrons moves diffusively rather than ballistically, the repulsive Coulomb interaction is enhanced and, of course, tends to inhibit Cooper pair formation (b) at least for elastic mean free path $l \ll d$ (thickness of film) (so that the system is “locally” 3D) the depression of T_c relative to its value T_{c0} for the pure material is given in perturbation theory by the formula

$$\ln(T_c/T_{c0}) = \frac{-e^2}{6\pi^2\hbar} g_1 R_{\square} (\ln 1/T_c\tau)^3 \quad (3)$$

where τ is the normal-state elastic scattering time, R_{\square} is as above the normal-state sheet resistance and g_1 is a dimensionless parameter characterizing the screened Coulomb interaction. Evidently this formula does not lead to suppression of T_c to zero for any value of τ , so it cannot by itself describe the S-I transition. A more sophisticated renormalization-group calculation, however, does predict that superconductivity vanishes (and hence the system presumably reverts to the insulating state) when

$$(e^2/2\pi\hbar)R_{\square} \sim (1/\ln T_{c0}\tau)^2 \quad (4)$$

Note that since R_{\square} and τ are independent variables, the S-I transition does not occur at a universal value of R_{\square} .

The point to stress about the above approach is that it is essentially a theory of the conditions under which *local* Cooper pairing is or is not favorable. The thermodynamics is thus very similar to that in the original BCS theory, and in particular it unambiguously predicts that the single-particle energy gap Δ (which can be measured, at least in principle, by e.g. tunnelling or specific heat measurements) should scale with T_c just as in that theory and thus approach zero as we approach the S-I transition from the S side. Another way

⁷Finkelstein, Physica B **197**, 636 (1994).

of putting it is that in this approach the reason superconductivity is destroyed by disorder is that the *amplitude* of the order parameter $\Psi(r)$ is driven to zero, not that its phase fluctuates as in the KT transition.

An alternative, very different approach to the S-I transition in thin dirty metallic films is based on the idea that the superconducting order parameter is locally defined throughout the whole of the parameter space, but that while in the S phase it possesses long-rang phase coherence (LRO), on the I side of the transition the phase fluctuates so wildly (due to quantum, not thermal effects) that no LRO is possible and the system becomes insulating. Thus, the S-I transition qualitatively resembles the KT transition which, as we saw in lecture 12, is believed to occur in helium films and also in some (less dirty) 2D metallic films. However, in the present case the transition does not occur in a given film as a function of temperature, but rather occurs at or close to zero T as a function of parameters such as thickness and sheet resistance (i.e. disorder). It is thus an example of a *quantum phase transition (QPT)*, that is a phase transition that occurs at zero temperature as one or more parameters of the system are varied. We need, therefore, to explore the general theory of such transitions a little.⁸

QPT's can of course occur in any number of (spatial) dimensions, including zero (see below), so the problem they pose is not restricted to $d = 2$. Exact analytic results can be obtained in only a small number of cases, of which the best known is the (quantum) 1D Ising model in a transverse field. Where no analytic solution is available, the theory of QPTs relies heavily on mapping the d -dimensional quantum problem to an equivalent $d + 1$ -dimensional classical problem and then using the large body of results which have been obtained over the last 40 years concerning (classical) second-order phase transitions. Let's illustrate this procedure with a particularly simple example, namely a 2D Josephson junction array. We already looked briefly, in lecture 12, at the experimental results on this system in a particular limit (see below) and saw that they are consistent with the hypothesis that it sustains a KT transition at a finite temperature determined by the strength $E_J (\equiv J)$ of the Josephson coupling. A point that will be essential in the arguments below is that the important length scales involved in the KT transition (ξ^\pm) diverge in the limit $T \rightarrow T_{KT}$, which means that the "bending" of the OP phase in the relevant region is very gentle and the system cannot tell the difference between the actual Hamiltonian, which refers to discrete grains, and a simple continuum such as would be realized by a ^4He film. (Technically, these two different systems are said to belong to the same "universality class.")

The Hamiltonian of the Josephson junction array was taken in lecture 12 to consist only of a potential energy that depends on the relative phase of the OP on neighboring

⁸For a readable short review of QPT's, see Sondhi et al., RMP **69**, 315 (1997), or for a more extended discussion, S. Sachdev, *Quantum Phase Transitions*.

dots:

$$V = E_J \sum_{ij=n.n.} \cos(\varphi_i - \varphi_j) \quad (5)$$

where E_J is (in zero external magnetic field) a positive constant. For small phase drops $\varphi_i - \varphi_j$ between nearest neighbors, (5) is equivalent to a continuum model with an equivalent superfluid density $\rho_{s0}(T)$ given by

$$\rho_{s0}(T) = (2m/\hbar)^2 E_J \quad (6)$$

(note that the spacing of the dots does not enter). The order of magnitude of T_{KT} is therefore⁹, perhaps unsurprisingly, E_J . Note that this means that for $T \sim T_{KT}$ the mean-square value of $\varphi_i - \varphi_j$ is not very small, so that the problem must be handled self-consistently and the difference between ρ_{s0} and ρ_s may be substantial. However, this does not affect the crucial point that the critical behavior at T_{KT} is sensitive only to configurations in which $|\varphi_i - \varphi_j| \ll 1$.

The above discussion is appropriate to “classical” Josephson junction arrays, by which we mean arrays in which the dots are large enough, and the inter-dot Josephson coupling E_J strong enough, that $E_J \gg E_c$, where $E_c \equiv \frac{(2e)^2}{2C}$ is the capacitive energy necessary to transfer a Cooper pair from one dot to its neighbor. However, for small dots and/or weak Josephson coupling E_c may be comparable to or even much larger than E_J . What is the form of the relevant term in the Hamiltonian? There are a number of ways of deriving it, of which the simplest is probably to note that the voltage V_{ij} between a pair i, j of neighboring dots is given in terms of the time derivative of $\Delta\varphi_{ij}$ by the second Josephson equation

$$V_{ij} = \frac{\hbar}{2e} \frac{d}{dt}(\Delta\varphi_{ij}) \quad (7)$$

Thus, for a single pair of dots ($\Delta\varphi_{ij} \rightarrow \varphi$) the total Hamiltonian can be written ($\Phi_0 \equiv h/2e =$ superconducting flux quantum)

$$\hat{H} = \frac{1}{2}CV^2 - E_J \cos \varphi = \frac{1}{2}C \left(\frac{\Phi_0}{2\pi} \right)^2 \dot{\varphi}^2 - E_J \cos \varphi \quad (8)$$

To express \hat{H} in terms of φ and its canonically conjugate momentum p_φ , we define $p_\varphi \equiv \partial T / \partial \dot{\varphi} = C(\Phi_0/2\pi)^2 \dot{\varphi}$, which from (7) is just $(\Phi_0/2\pi)$ times $CV = Q$, the charge transferred across the junction; the “kinetic energy” T (the first term in (8)) is then just $p_\varphi^2 / (2C (\Phi_0/2\pi)^2) \equiv Q^2 / 2C$. However, it will actually be more convenient for our purposes to keep the Hamiltonian in the (non-canonical) form (8), i.e. to regard it as a function of φ and $\dot{\varphi}$ rather than of φ and p_φ .

⁹Of course E_J may itself be temperature-dependent, but such dependence is likely to be on the scale of the bulk 3D BCS transition temperature T_{c0} , which in real life is typically $\gg E_J(T=0)$; thus it may safely be neglected in the present context.

The Hamiltonian (8) of a single pair of dots connected by a Josephson junction is nothing but that of a simple quantum pendulum, with the quantity E_J corresponding to the gravitational potential and the quantity $C(\Phi_0/2\pi)^2$ playing the role of moment of inertia. A simple analysis shows that in the limit $E_c \ll E_J$ the groundstate of the system has φ close to zero and the low excited states are harmonic-oscillator-like with frequency $\omega_J \equiv \sqrt{E_J E_c}/\hbar$, while for $E_c \gg E_J$ the groundstate is approximately the state of zero angular momentum (i.e. uniform with respect to φ) and excited states are rotor-like. For intermediate values of $K \equiv E_c/E_J$ the problem can be solved to high accuracy by numerical methods, and it is found that the crossover is smooth as a function of K .

Let's now return to our 2D array. From the above considerations, the Hamiltonian, including capacitance effects, can be written in the form*

$$\hat{H} = \sum_{ij=n.n.} \left\{ \frac{1}{2} C (\Phi_0/2\pi)^2 \dot{\varphi}_{ij}^2 - E_J \cos \varphi_{ij} \right\} (\varphi_{ij} \equiv \varphi_i - \varphi_j) \quad (9)$$

Let us consider the partition function

$$Z(\beta) \equiv \text{Tr} \exp -\beta \hat{H} \quad (\beta \equiv 1/k_B T) \quad (10)$$

In principle, if we can write Z as a function not only of β but of arbitrary “external source” terms, then by differentiating with respect to these source terms we can obtain the expectation value, in thermal equilibrium, of an arbitrary (time-independent) operator. So it contains complete information about all the static properties of the system.

Before proceeding, we need to dispose of one technical point: Let's write $\varphi_{ij} \equiv \varphi_i - \varphi_j$, so that the “kinetic energy” term in (9) becomes

$$T = \frac{1}{2} C (\Phi_0/2\pi)^2 \sum_{ij=nn} (\dot{\varphi}_i^2 + \dot{\varphi}_j^2 - 2\dot{\varphi}_i \dot{\varphi}_j) \quad (11)$$

The last term has the form

$$\text{const.} \left(\sum_i \dot{\varphi}_i \right)^2 = \text{const.} \left(\sum_i V_i \right)^2 \quad (12)$$

It is clear that this term can always be eliminated by a suitable choice of the zero of voltage (electrochemical potential). Hence we can legitimately write

$$T = \frac{1}{2} \tilde{C} (\Phi_0/2\pi)^2 \sum_i \dot{\varphi}_i^2 \quad (13)$$

*We assume for simplicity that the dots form a square lattice and that further-neighbor interactions are negligible. In the more general case C is replaced by a matrix C_{ij}

where $\tilde{C} = 2zC$, $z =$ number of nearest neighbors. We will use this form rather than that in (9) below.

It is well known that whether the system is quantum or classical, the partition function can be written in path-integral form in “imaginary time” τ (we suppress any dependence on external source terms):

$$Z(\beta) = \prod_i \int \mathcal{D}\varphi_i(\tau) \exp - \int_0^{\beta\hbar} \mathcal{H}[\varphi_i(\tau)] d\tau/\hbar \quad (14)$$

Formula (14) is exact. When the temperature is high compared to the relevant characteristic frequencies $\bar{\omega}$ of the system, so that $\bar{\omega}\tau \ll 1$ over the whole range $0 < \tau < \beta\hbar$, there is no “time” for fluctuations of φ_i as a function of τ to occur, and they may be set equal to their values at say $\tau = 0$; the “kinetic energy” term (the first term in (9)) now contributes at most an uninteresting constant, and we can replace (14) by the classical expression

$$Z_{cl}(\beta) = \prod_i \int \mathcal{D}\varphi_i \exp -\beta V\{\varphi_{ij}\} \quad (15)$$

Now comes the nontrivial “trick”. We insert the full quantum expression (9), with the replacement (13), into (14) and *postulate* that for the purposes of deriving the critical behavior the important imaginary-time trajectories have the property that the “gradients” of the φ_i are “small,” not just in space *but also in (imaginary) time*, in the sense that $\varphi_i(\tau + \Delta\tau) - \varphi_i(\tau) \ll 1$ when $\Delta\tau$ is an appropriate small time interval (to be defined below). We then discretize the integral in the exponent of (14), replacing the continuous variable τ by a set of discrete values $\tau_n \equiv n\Delta\tau$, so that considering the $n, n + 1$ pair of times and the phase φ_i the “kinetic energy” T of eqn. (13) is given by

$$\begin{aligned} T &= (\text{const.}) (\Delta\tau)^{-2} (\varphi_i(\tau_{n+1}) - \varphi_i(\tau_n))^2 \\ &\cong \text{const.} - \text{const.} \cdot 2 (\Delta\tau)^{-2} (\cos\{\varphi_i(\tau_{n+1}) - \varphi_i(\tau_n)\}) \end{aligned} \quad (16)$$

Hence the exponent S of the functional integral in (14) takes the discretized form

$$\begin{aligned} S &= \hbar^{-1} \left\{ \sum_n \sum_i \left(-\frac{1}{2} \tilde{C} (\Phi_0/2\pi)^2 (\Delta\tau)^{-1} \right) \cos[\varphi_i(\tau_{n+1}) - \varphi(\tau_n)] \right. \\ &\quad \left. - E_J \Delta\tau \sum_{ij=\text{n.n.}} \cos[\varphi_i(\tau_n) - \varphi_j(\tau_n)] \right\} \end{aligned} \quad (17)$$

It is clear that the structure of the two terms is exactly the same. In fact, if we make the specific choice $\Delta\tau = \hbar/\sqrt{E_J E_c}$ where E_c now $\equiv (2e)^2/2\tilde{C}$ and remember that $\Delta\tau = \beta\hbar/L_\tau$ where L_τ is the number of “time” steps, we can write

$$S = -\frac{1}{K} \left\{ \sum_{i,n} \cos(\varphi_i(\tau_{n+1}) - \varphi(\tau_n)) + \sum_{ij=nn} \cos(\varphi_i(\tau_n) - \varphi_j(\tau_n)) \right\} \quad (18)$$

where

$$K \equiv (E_c/E_J)^{1/2} \quad (19)$$

We can now change the notation so that $\varphi_i(\tau_n) \rightarrow \varphi_{in}$ and the functional integral (14) then takes the discretized form

$$Z(\beta) \rightarrow \prod_{i,n} \int \mathcal{D}\varphi_{i,n} \exp -\frac{1}{K} \left\{ -\sum_{in} \cos(\varphi_{i,n+1} - \varphi_{i,n}) - \sum_{ij=\text{n.n.}} \cos(\varphi_{i,n} - \varphi_{j,n}) \right\} \quad (20)$$

But this is nothing but the *classical* partition function of a 3-dimensional array of Josephson junctions, with the quantity K playing the role of $k_B T/E_J$ and the physical temperature β now playing the role of the “width” L_τ (number of sites) in the third dimension! Thus we can immediately read off some of the properties of the 2D zero-temperature QPT from our knowledge of the behavior of the 3D *classical* phase transition of the XY-model (since this is just the model described by (18)). In particular, since we know that in the thermodynamic limit the 3D XY model has a phase transition from an ordered to a disordered state at a value of $k_B T/E_J \sim 1$, it follows that at zero temperature ($L_\tau \rightarrow \infty$) the 2D quantum model should have a similar transition at a value of $K \sim 1$. (However, we cannot necessarily identify the constant factors in the two cases, since these may be sensitive to short-range effects for which the replacement (16) need not be valid.) Moreover, as K approaches the critical value K_c , we expect a divergence of the characteristic length scale which is identical in form to that of the 3D XY model for $T \rightarrow T_c$; in fact, if

$$\xi_{3D}(T) \rightarrow |T - T_c|^{-\nu} \quad (21)$$

then at the 2D QPT $\xi_{2D}(K)$ should diverge in the same way:

$$\xi_{2D}(K)_{(T=0)} \rightarrow |K - K_c|^{-\nu}. \quad (22)$$

For the 3D XY model both theory and experiment (on superfluid ^4He) suggest that ν is close to $2/3$.

It should be cautioned that we could make this specially simple correspondence between the zero-temperature d -dimensional QPT and the (finite- T) $d+1$ -dimensional classical one only because we could argue that for the trajectories which play the crucial role in the transition the form of the “time-axis” interactions is approximately the same as the “space-axis” ones, namely $\cos(\varphi_n - \varphi_{n+1})$. This feature is not generic to QPT’s; it is essentially equivalent to the statement that for the classical phase transition the exponents of the divergent correlation length ξ and the divergent correlation time ξ_τ , which are conventionally called ν and $z\nu$ respectively, i.e.

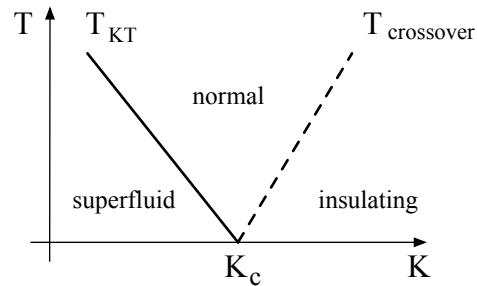
$$\xi \sim |T - T_c|^{-\nu} \quad (23a)$$

$$\xi_\tau \sim |T - T_c|^{-z\nu} \quad (23b)$$

are the same, i.e. $z = 1$. If the classical $d + 1$ -dimensional phase transition does not satisfy this condition the relation to the quantum d -dimensional transition is more complicated.

For values of K much less than K_c our 2D Josephson junction array is clearly qualitatively similar to the 3D array well below T_c , i.e. it possesses LRO (remember, we are for the moment at $T = 0$, so the HMW theorem does not forbid this!) and behaves as a superfluid. What is the nature of the phase which occurs at values of K greater than K_c ? Here the analogy with the classical transition is less useful, since the latter automatically occurs at finite T where there are already many excitations. However, explicit consideration of a similar model, the Bose alkali gas in an optical lattice, suggests that this phase has a definite number of particles (in our case Cooper pairs) on each lattice site (dot); unless we are close to K_c such a phase should have an energy gap of the order of the capacitance energy E_c , and thus at zero temperature should be insulating. Thus the transition we have discovered in the 2D Josephson junction array might plausibly be regarded as corresponding to the superconductor-insulator transition observed experimentally in thin films, at least those of sufficient granularity.

What can we say about the behavior of the system at nonzero temperatures and K close to K_c ? Actually, scaling considerations help us quite a bit. Consider first the general nature of the phase diagram. We know that for $K \ll K_c$ we have (except very close to zero temperature) a 2D *classical* XY model, which has a KT transition, at a temperature $\sim E_J$, from a phase with (topological) LRO to a “normal” phase. So there



must be a phase boundary (locus of second-order transitions) in the K - T plane. Also, we know that at $T = 0$ the “disordered” phase is an insulator while at high T (crossing over from the KT transition) it is a normal “liquid”; hence there must be a rough separatrix in the K - T plane between normal-metallic and insulating behavior; however, this could be a smooth crossover since no symmetry is broken on either side and the resistivity is expected to be continuous. Qualitatively, we should expect this crossover to occur at a temperature $T(K) \sim \Delta(K)$ where Δ is the excitation gap at zero temperature. Anticipating the result that $\Delta(K) \rightarrow 0$ for $K \rightarrow K_c$ (see below) we therefore find the phase diagram should look *qualitatively* like shown in the figure. Let’s try to be a bit more quantitative. We recall that the length along the “time” axis, L_τ , is proportional to the inverse temperature β . Thus, as temperature increases, L_τ decreases. We should expect that the behavior changes qualitatively at the point where L_τ becomes of the order of ξ_τ ($\sim \xi$ for our ($z = 1$) case) for the 3D problem; beyond this, the classical “3D” problem becomes that of a “slab” of thickness $< \xi_\tau$, whose behavior we should expect to be essentially 2D. This criterion gives a

critical temperature (for $K < K_c$) $T_{KT}(K)$ of order $\xi_{3D}^{-1}(K)$, and since $\xi_{3D} \sim |(K - K_c)|^{-\nu}$ we find

$$T_{KT}(K) = \text{const.}(K_c - K)^\nu \quad (24)$$

A similar argument on the insulating side shows that the crossover temperature (which is only defined up to an order of magnitude) should satisfy

$$T_{\text{crossover}}(K) = \text{const.}(K - K_c)^\nu. \quad (25)$$

Hence the phase diagram is roughly as shown in the figure: since for our case $\nu < 1$, there is a cusp at the QCP $T = 0, K = K_c$.

Although we have worked out this scenario for the particular case of a 2D JJ array (which happens to correspond to $z = 1$), it should be qualitatively valid for an arbitrary second-order QPT, with the proviso that while the separatrices T_0 still satisfy the criterion $L_\tau \sim \xi_\tau$, the quantity ξ_τ in general scales like $|K - K_c|^{-z\nu}$ (where K is now a general control parameter), and so we have

$$T_0(K) \sim \text{const.} |K - K_c|^{z\nu}. \quad (26)$$

We can actually go somewhat further. Consider a physical quantity such as the dc resistivity R , as a function of K and T . Since this corresponds to $k = \omega = 0$, it does not by itself define a length or time scale. Hence it must be a function only of the ratio of the two “large” lengths in the problem, namely ξ_τ and L_τ . Since $L_\tau \sim T^{-1}$ and $\xi_\tau \sim |K - K_c|^{-\nu z}$, it follows that ($\delta \equiv K - K_c$)

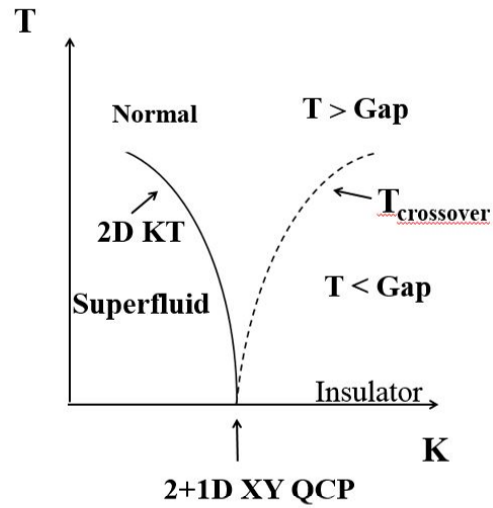
$$R(\delta, T) = R_c f(\delta \cdot T^{-1/\nu z}) \quad (27)$$

where R_c may be taken as the resistivity exactly at the QCP. Of course, the function f could have a different form on the two sides of the transition.

This result should hold irrespective of the nature of the control parameter. It turns out that a similar scaling relation can be derived for the dependence on the electric field applied across the sample

$$R_c(\delta, E) = R_c f(\delta \cdot E^{-1/\nu(z+1)}) \quad (28)$$

(though see below for a caveat)



We now return to the experimental situation: see Lin et al., ref. cit., section 4. The S-I transition in thin metallic films can be tuned by thickness, disorder (if we believe that is a different variable), magnetic field or electric field (though see below on the last). Some of the data, for example those on amorphous Bi grown on an $\alpha - Ge$ substrate, fit the scaling relation (27) beautifully; see Lin et al, fig. 3, where the product νz is found to be $\cong 1.2$. However, experiments on the same samples which kept the thickness constant and varied the magnetic field also found excellent scaling behavior, but with $\nu z \cong 0.7$, suggesting that the two transitions may not be in the same universality class. The scaling with electric field was also measured, and originally it was thought* that by comparing this to eqn. (28) one could extract also the combination $\nu(z + 1)$ and thus obtain ν and z independently; however, subsequent experiments suggested that this conclusion may be vitiated by heating effects produced by the electric field (see Lin et al. section 5). Thus the situation remains somewhat murky...

A system which is somewhat similar to thin films, but should be more extremely 2-dimensional, is the “interfacial 2D electron gas” formed by electrons trapped at the interface between two dielectrics (insulators): the latter can be both band insulators (e.g. $LaAlO_3/SrTiO_3$) or a Mott insulator and a band insulator (e.g. $LaTiO_3/SrTiO_3$)*. In this case the areal electron density, which can be controlled by applying a gate voltage to one of the dielectrics, is the control parameter δ for the transition (at nonzero T) from a normal to a superconducting state, and the experiment on LAO/STO are consistent with the scaling law $T_c(\delta) \propto \delta^{z\nu}$ with $z\nu \cong 2/3$. Moreover, the resistance in the normal state agrees well with the KT prediction $R \propto \exp[-b/(T - T_c(\delta))^{1/2}]$. On the other hand, experiments by a different group on LTO/STO also indicate scaling behavior, but with $\nu z = 3/2$; the reason for this difference is not currently well understood. One more related system is thin (one-unit-cell thick) cuprate layers doped with an ionic liquid gate (a method which permits much higher doping levels than that by ferro- or strongly paraelectric materials); the interpretation of the raw data is in this case currently controversial, see Lin et al., ref. cit., section 8.

*see Goldman, Physica E **18**, 1 (2008)

*See Lin et al., Physica C **514**, 130 (2015)