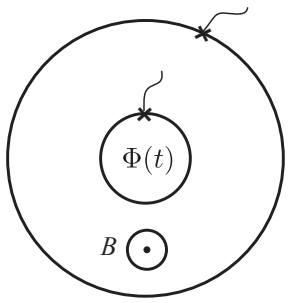
The integral QHE: Topological considerations, edge states

Generally speaking, the occurrence of integer or rational-fraction quantum numbers in QM is a result either of some *symmetry* of the problem, or of *topological* considerations, or of both. However, as emphasized by Thouless,[1] those numbers that are the consequence of topology are usually much more stable against small perturbations than the symmetry-derived kind. To illustrate this point, consider the (meta) stability of a circulating-current state of ^{4}He in an annulus. If the annulus were exactly cylindrical in shape (i.e. possessed exact invariance under rotation around its central axis), then angular momentum would be a good quantum number and independently of any He-He interactions we could attribute the metastability of the rotating current to its conservation. However, in real life there will certainly be small departures from cylindrical symmetry (both static and dynamic) and those will spoil the conservation of angular momentum. On the other hand, provided only that the amplitude of the order parameter is everywhere nonzero on some path around the ring, then as we saw in lecture 9, we can define the "winding number" $n \equiv \oint \boldsymbol{\nabla}\varphi \cdot \boldsymbol{dl}$; this will be conserved irrespective of the detailed geometry of the ring, provided only that we can neglect exponentially rare fluctuations of the Langer-Fisher type, and it is this feature, not the symmetry, which is generally believed to play the crucial role in stabilizing the circulating-current state.

Fig. 1



In the case of the QHE (whether integral or fractional), the very high degree of robustness of the Hall resistance against small changes (and in some cases even large ones) of materials parameters, geometry, etc., suggest very strongly that the origin of the affect is topological, and essentially all explanations in the literature rely on this this feature, at least by implication. I start with what is probably the simplest approach, due to Laughlin and Halperin. For the moment I assume a single "valley" and spin index, so that the filling factor ν is just the number of electrons/Landau level.

Consider a simple Corbino-disk geometry (fig. 1), with current leads attached to the inner and outer edges, to which is applied a uniform field B *plus* an "Aharonov-Bohm" flux, which may vary in time. The total flux through the hole, or equally through any circular orbit within the disk, is thus an arbitrary function of time.

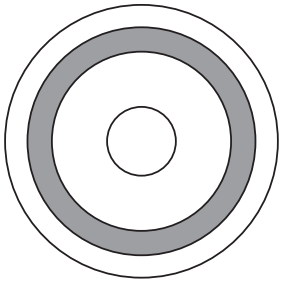
As in lecture 16, we may choose a radial gauge and write the energy eigenfunctions of a given LL n for fixed flux in the form $\psi_{nl}(r,\varphi) = \exp il\varphi\, H_n(r - r_l)$ where r_l is the radius of an orbit enclosing exactly l quanta of flux. A point to notice is that the quantity r_l will in general depend on the AB flux Φ. These eigenstates carry no current, in either the angular or the radial direction (in the angular case, the "canonical" angular momentum $l\hbar$ is just canceled, when we average over the radial wave functions, by the "gauge" term in

[1] J. Math. Phys. **35**, 5362 (1994).

the expression for the current).

Now imagine that we slowly increase the AB flux through the hole, thereby generating an emf $V = -\partial\Phi_{AB}(t)/\partial t$ around the disk. Assume that the single-electron wave functions evolve adiabatically, i.e. so that they are solutions to the TISE for the "instantaneous" value of Φ_{AB}; then it is clear that the guiding center of each state, r_l, will move outwards. In fact, when we have increased the flux by one unit $h/e \equiv \varphi_0$, each state will have exactly replaced its outer neighbor (and one state will have been added at the inside edge and disappeared at the outer edge). Now, if (and only if!) each state contains an integral number n of electrons, then the net result will have been the transport of n electrons from the inner to the outer edge of the disk. This gives a current ne/τ, where $\tau \equiv \varphi_0(d\Phi_{AB}/dt)^{-1} \equiv \varphi_0 V$, and thus a Hall conductance $ne/\varphi_0 \equiv ne^2/h$.

Fig. 2



So far, so good, but all we have done is to recover the "naive" result that *if* there are exactly n electrons for l-states, i.e. exactly n LL's are filled, then we get a conductance ne^2/h. We still have to explain the existence of finite plateaux. The explanation, surprisingly, lies in the existence of disorder and hence of localized states. As usual in the theory of the QHE, we argue that since the experimentally observed effect is essentially independent of geometrical details we may choose any convenient geometry, and following Halperin we imagine that the disorder is confined to a section of the disk of intermediate radius, with two ideal "guard rings" inside and outside it (see Fig. 2). Then, as regards the guard rings, the states are just as previously, and in particular a change in the AB flux results in the motion of exactly one state across each of them. Consider now the situation in the disordered region. Here, according to the arguments of lecture 16, we expect to find within a given LL both localized states (in which the electrons circulate around "hills' or "valleys" of the potential) and "extended" states, which extend right around the disk; in the latter the behavior is qualitatively similar to that in the guard rings (and in particular there is no angular current in an energy eigenstate). A crucial consideration is that, barring some rather pathological cases[2], the eigenstates at a given energy either wind right around the disk or enclose a finite number of isolated hills or valleys; the latter case corresponds to the edges of the band and the former to the middle. Thus, it follows that (excluding pathologies) the "band" corresponding to a given LL separates into three distinct regions: a region at the upper end where all orbits are localized and any electrons in them circulate (clockwise) around "peaks," a middle region where all the states are extended around the whole disk, and a lower region where the states are all localized and any electrons in them circulate (anticlockwise) around "valleys" (troughs) in the local potential. It is intuitively

[2]For example involving "inland seas."

clear that the (unique) energy of the extended states in the guard rings lies somewhere in the "extended" region of the disordered-region spectrum.

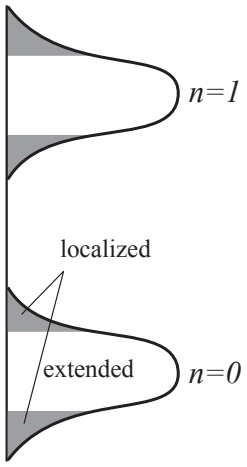


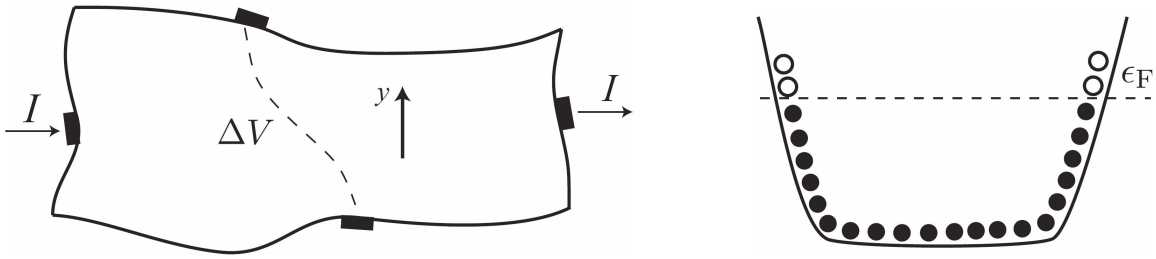
Fig. 3

It is now obvious that the crucial question (within a single-electron picture) is: Where does the Fermi energy (chemical potential) lie? If it lies in the *extended* region, then the states of the guard rings will not be filled with an integral number of electrons, and the transported current will not in general correspond to an integral number of electrons transported per unit change of the AB flux. If on the other hand the chemical potential lies anywhere in the *localized* region[3], then the extended states of a given LL will be either all full or all empty and our argument regarding the guard rings goes through: for each integral change of the AB flux, n electrons are transported across both the inner and the outer guard rings. A final, vital step in the argument is that since the localized states are not affected by the AB flux, their energies and thus their occupation factors cannot change; thus, for any given electron transported across the inner guard ring into the disordered region, one must leave this region and cross the outer guard ring. Thus the total current across the disk is exactly $(ne^2/h)V$, where n is the number of LL's whose *extended* regions are occupied, so $\Sigma \equiv I/V = ne^2/h$, the IQHE. Note that the fraction of states in the disordered region (or in a more realistic model, in the whole system) affects only the length of the plateau, not the quantized conductance itself. Also note that the above argument gives no particular reason to believe that all the plateaux have the same length: the localized fraction could well itself be a function of n.

While the Corbino-disk geometry provides (in my opinion) the simplest argument, it has the drawback that the Hall resistance cannot be easily measured in it. An alternative approach to the IQHE relies on the concept of *edge states*. For this we need to consider a different geometry (fig. 4a). In this case we may imagine that near the walls of the physical sample the potential rises drastically, so that viewed in cross-section the profile $V(y)$ is roughly as indicated (fig. 4b). (There must also be some irregularities, not shown, in the potential in the bulk of the system in order to get the finite density of localized states, which is necessary to produce a finite width for the plateaux.)

Consider initially the case $\Delta V = 0$, (so that $V(y)$ is entirely due to electrostatic affects within the sample itself). We consider e.g. the $n = 0$ LL and assume that ϵ_F is well above the "bulk" energy ($\sim \hbar\omega_c$), so that all the extended states in the bulk are filled. As we approach the edge, the energy rises sharply (recall from lecture 16 that $E_n = \hbar\omega_c + V(y_n) + \frac{m}{2}(E/B)^2$), so that *in this region* there are extended states (which circle the

[3]Or of course in the "gap" between LL's, though this only happens for discrete (integral) values of the filling factor.



(a) (b)

Fig. 4

sample) at the Fermi level.

Let's consider the current carried by a given state at the point x, y, for the moment ignoring any spatial variation of the "drift" term $\frac{1}{2}m(E^2/B^2)$ in the energy. From the results of lecture 16, the velocity of the state is given by $\mathbf{v} = \boldsymbol{E}(\mathbf{r}) \times \mathbf{B}/B^2$. Since the transverse extent of the wave function is just the magnetic length and independent of position on the contour, the probability density is constant and equal to L^{-1}, when L is the length of the contour. Hence (cf. lecture 16) the current I_n carried is (eE/BL) (along the contour).

Next we ask: what is the distance Δy to the next state (in the next allowed volume of the guiding-center locus)? We know that the total area "occupied" by the state (that is, the inverse of the DOS/unit area) is just φ_0/B, and this condition is satisfied if we assume that the distance between allowed orbits is independent of position on the orbit and equal to φ_0/BL. (If we were to assume that the inter-orbit distance varies along the orbit, we would get a value for the current I_n which is itself position-dependent, which cannot be the case for an energy eigenstate). Consequently, we get a relation between I_n and Δy,

$$I_n = eE\Delta y/\varphi_0 \tag{1}$$

But $eE\Delta y$ is simply the difference in energy between the levels E_n and E_{n+1}, so we finally get

$$\boxed{I_n = \frac{e}{h}(E_{n+1} - E_n)} \tag{2}$$

(This formula is actually valid for the localized states as well as the extended ones, but in the former case the resultant current does not contribute to the measured Hall effect.)

It is possible to derive eqn. (2) by an alternative method, which makes it clear that the neglect of the drift term in the energy is not essential (Prange et al., p. 79): we imagine changing the "single-valuedness" boundary condition around the orbit so that $\psi(x + L) = \exp 2\pi i\alpha\, \psi(x)$. (This can be achieved, physically, by imposing a flux $\alpha\varphi_0$ and

redefining ψ so as to get rid of the gauge term in the KE.) Then it is easy to show that

$$\hat{I}_n = \frac{e}{h}\frac{\partial \hat{\mathcal{H}}}{\partial \alpha} \tag{3}$$

Taking the expectation value and using the Feynman-Hellman theorem gives $I_n = -(e/h)\times(\partial E_n/\partial\alpha)$. When we vary α from 0 to 1, the effect is to shift each allowed orbit to the neighboring value (cf. the earlier discussion of the Laughlin-Halperin argument). Thus, integrating from 0 to 1, we arrive again at eqn. (2).

It is clear that if we sum eqn. (2) over the orbits of a given LL which lie between y and $y + \Delta y$ (where Δy is now arbitrary) we obtain for the total contribution in ΔI to the current

$$\Delta I = \frac{e}{h}\big((\mu(y+\Delta y) - \mu(y)\big) \equiv \frac{e}{h}\Delta\mu \tag{4}$$

where $\mu(y) \equiv -eEy + \frac{1}{2}m(E/B)^2$ is the electrochemical potential. In the bulk of the system (though not necessarily close to the boundaries) this can usually be safely equated[4] to the usual electrostatic potential, so we get $\Delta I = (e^2/h)\Delta V$ for each filled LL, or for the conductivity

$$\sigma_{xy} = ne^2/h \tag{5}$$

where n is the number of filled LL's. This is of course just the IQHE.

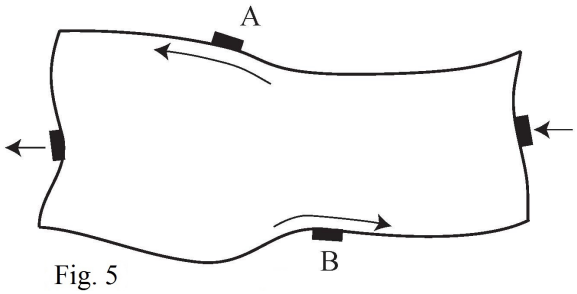


Fig. 5

What happens near the walls? Let's consider fig. 5 and continue to assume that the electrochemical potential μ is constant around the edges (e.g. detach the voltage and current leads). Then, if μ is measured relative to the floor of the potential, it is easy to see that there will be a current of magnitude $e\mu/h$ running around the edge of the sample. Since from (4) we can obtain this result separately for the top and bottom edges, it is clear that attaching current leads (but no voltage) will not change this result (since current conservation must be maintained).

Now suppose we apply an electrochemical potential difference $\Delta\mu$ (or, what is equivalent under most circumstances, a voltage $\Delta V \equiv e\Delta\mu$) between the voltage leads A and B. It is now clear that the quantity $\Delta\mu_{A,b}$ (the electrochemical drop between lead A and the bulk) will be increased and $\Delta\mu_{B,b}$ decreased. Consequently, more current will flow along the top edge (say to the left) and less along the bottom edge (to the right). The total current that flows in and out of the system through the current leads is, for each filled LL,

$$I = \frac{e}{h}(\Delta\mu_{A,b} - \Delta\mu_{B,b}) = (e^2/h)V \tag{6}$$

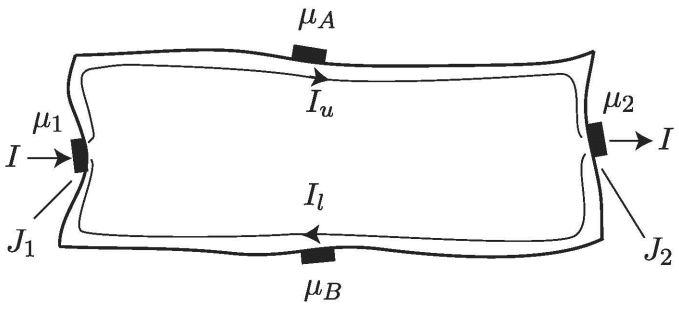


Fig. 6

again giving the IQHE. It is therefore clear that the Hall current flows entirely through the "edge" states, while the bulk of the sample remains inert.

It is interesting to make contact with the general theory of the conductance of 1D systems, due to Landauer, Büttiker and others[5]. In this theory one classifies the electron states incident on a given barrier into "channels," with each channel characterized by a different set of "transverse" quantum numbers. The conductance due to a given channel is then simply $(e^2/h) \times T)$, where the dimensionless quantity $T \equiv 1 - R$ is the transmission coefficient for this channel through the barrier (R is the corresponding reflection coefficient); the total conductance is just the sum of the conductances of the individual channels. In the present application (see fig. 6), the different channels correspond to the different occupied Landau levels; the "barriers" are the interfaces J_1, J_2 between the quantum Hall system and the *current* leads (there is no transmission, by hypothesis, into the voltage leads). The electrochemical potential μ is constant at that of the voltage lead μ_A along the top edge and constant at the different value μ_B along the bottom edge, changing between these values over some microscopic distance close to J_1 and J_2. Consider a given channel (Landau level). If $T_i(R_i)$ is the transmission (reflection) coefficient of junction i, then (for example) the current I_u has two contributions: that which flows in across J_1 from lead 1 and is transmitted, and that which flows in along the bottom edge and is "reflected" (i.e. does not exit through lead 1; since it cannot reflect into the bottom edge, it must continue as part of the upper-edge current I_u). At junction J_2, the only contribution to I_l is the part of I_u which is "reflected." Since we have according to our previous argument $I_u = (e/h)(\mu_A - \mu_b)$, $I_l = (e/h)(\mu_B - \mu_b)$, and from current conservation we have for the total current I flowing through the system $I = I_u - I_l$, we immediately recover the result that each channel contributes a conductance $I/V_H = e^2/h$, so $\Sigma_{\rm H} = ne^2/h$ where n is the number of Landau levels, as obtained previously. Also, by considering the process

[4]If one objects that this is not necessarily true to one part in 10^8, the answer is that the conductivity should strictly speaking be defined as the ratio of current to electrochemical potential rather than simply to voltage.

[5]This discussion follows that of Yoshioka section 3.2.3, with minor variations.

of transmission/reflection at the junctions J_1, J_2 we can also obtain the quantitites μ_1, μ_2 and so, if desired, relate I to the difference $\mu_1 - \mu_2$. According to Yoshioka (loc. cit., eqns. (3.21–4)) the result for a single channel is

$$I = \frac{e}{h} \frac{T_1 T_2}{1 - R_1 R_2} (\mu_1 - \mu_2) \tag{7}$$

However, it should be emphasized that the ratio $I/(V_1 - V_2)(V_i \equiv \mu_i/e)$ is *not* the longitudinal conductance of the Hall bar itself (which is still zero); rather, it represents the series *contact resistance* of the junctions J_1, J_2. Note that it is precisely at these junctions (and only these) that heat is dissipated; this is confirmed experimentally.

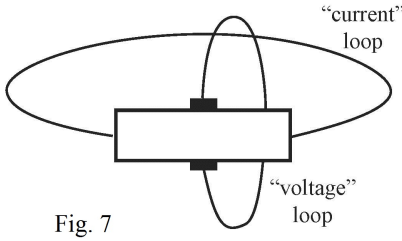


Fig. 7

The above considerations give a convincing physical picture of the origin of IQHE, but one might be a little worried that they rest on an independent-electron-type picture. Is it possible to obtain the effect more generally from the properties of the *many-body* wave function, without making the independent-electron approximation? The following elegant, if not 100% rigorous, argument is due to Thouless and co-workers[6]. Suppose that we take our Hall bar and join up both the sides and the ends by a loop of the same material, thus making as it were a fraction of a torus (fig. 7). We apply a time-dependent AB flux $\Phi_{\rm v}(t)$ through the loop attached to the voltage leads, and measure the current $I_J(t)$ around the loop connected to the current leads. The Hall conductance is the ratio $I_J/(d\Phi_{\rm v}/dt)$.

Since the perturbation caused by $\Phi_v(t)$ is of the form $\delta\hat{H} = -\hat{I}_{\rm v}\Phi_{\rm v}(t)$, we have

$$\Sigma_{\rm H} = {\rm Re}\left\{\frac{1}{i\omega} \ll I_{\rm v} : I_J \gg (\omega)\right\} \tag{8}$$

where $\ll A : B \gg$ is the standard linear response function. Writing out the latter in terms of matrix elements and using $I = \partial\hat{H}/\partial\Phi$, we get ($\varphi_J \equiv 2\pi\Phi_J/\Phi_0$, etc.)

$$\Sigma_{\rm H}(\varphi_j, \varphi_v) = i\hbar\, \langle 0|I_{\rm v}\frac{\mathcal{P}}{(E_0 - \hat{H})^2} I_J - I_J \frac{\mathcal{P}}{(E_0 - \hat{H})^2}\, I_{\rm v}|0\rangle \tag{9}$$

$$\equiv i\hbar\langle 0|\, \frac{\partial\hat{H}}{\partial\varphi_{\rm v}}\, \frac{\mathcal{P}}{(E_0 - \hat{H})^2}\, \frac{\partial\hat{H}}{\partial\varphi_J} - \frac{\partial\hat{H}}{\partial\varphi_J}\, \frac{\mathcal{P}}{(E_0 - \hat{H})^2}\, \frac{\partial\hat{H}}{\partial\varphi_{\rm v}}\, |0\rangle\Phi_0^{-2}$$

where $|0\rangle$ is the groundstate (with energy E_0) and $\mathcal{P}$ projects this off (so that the energy denominator is never zero). Now, it is easy to demonstrate that the change of the GSWF Ψ_0 with (any) φ_v is given by

$$\frac{\partial\Psi_0}{\partial\varphi_{\rm v}} = \frac{\mathcal{P}}{E - \hat{H}}\, \frac{\partial\hat{H}}{\partial\varphi_{\rm v}}\, |\Psi_0> \tag{10}$$

[6] see e.g. J. Math. Phys. **35**, 5362 (1994).

and so the above expression for Σ_H can be rewritten

$$\Sigma_{\rm H}(\varphi_J,\varphi_v) = i\hbar\,\Phi_0^{-2}\int d^N r\,(\partial\Psi_0^*/\partial\varphi_{\rm v}\,\partial\Psi_0/\partial\varphi_J - \partial\Psi_0^*/\partial\varphi_J\partial\Phi_0/\partial\varphi_{\rm v}) \tag{11}$$

Let us now integrate this relation over both φ_J and $\varphi_{\rm v}$ from 0 to 2π. The average conductance $\overline{\Sigma}_{\rm H}$is then given by

$$\overline{\Sigma}_{\rm H} = \frac{e^2}{h}\cdot\frac{i}{2\pi}\left[\int_0^{2\pi} d\varphi_J\int_0^{2\pi} d\varphi_{\rm v}\left\{\int d^N r(\partial\Psi_0^*/\partial\varphi_{\rm v}\cdot\partial\Psi_0/\partial\varphi_j - \partial\Psi_0^*/\partial\varphi_j\,\partial\Psi_0/\partial\varphi_{\rm v})\right\}\right] \tag{12}$$

An essential assumption, now, is that the groundstate returns to its original value (of course modulo $2n\pi$ in phase) when $\varphi_J \to \varphi_J + 2\pi$ and $\varphi_{\rm v} \to \varphi_{\rm v} + 2\pi$. Now with that constraint there is a general theorem that the quantity in $\left[\ \right]$ is 2π times an integer n which defines the "first Chern class" of the mapping $\varphi_J, \varphi_{\rm v} \to \Psi_0$, so we finally obtain

$$\overline{\Sigma}_{\rm H} = ne^2/h \tag{13}$$

(Thouless argues, in effect, that $\overline{\Sigma}_{\rm H}$ can be identified with $\Sigma_{\rm H}$, but in any case the above result guarantees the IQHE in an "average" sense.) The number n is essentially the number of times the phase of Ψ_0 "wraps" around the torus.

Before leaving the IQHE, one point should be emphasized: While we have concentrated on establishing that $\Sigma_{\rm H} = ne^2/h$, it is equally vital to the explanation of the experimental results that $\Sigma_{xx} = 0$ (if this were not so, it would be difficult to understand the independence of Σ_{xy} from the details of the geometry). In the "single-electron" picture given above, this effect is rather trivial since provided the extended and localized states occupy separate energy regions, it follows that whenever the Fermi energy lies in the "localized" region Σ_{xx} must be zero; since we have shown that this position of E_F is also a necessary and sufficient condition for quantization of Σ_{xy}, the two phenomena must always go together, as they of course do experimentally. It is an interesting question whether one could prove this more generally, perhaps by a slight extension of the Thouless argument.

What kinds of effect would we expect to destroy the IQHE? The most obvious is temperature, since when $k_{\rm B}T$ becomes an appreciable fraction of $\hbar\omega_c$ we expect that electrons would be excited (e.g.) out of the LLL into higher Landau levels and the condition of perfect filling of the extended states would no longer be realized (in fact this happens at the weaker condition $k_{\rm B}T \sim$ impurity bandwidth[7]). It turns out that the effect is also destroyed when we attempt to pass too large a current through the system: this is believed to be due to thermal "avalanching," cf. Yoshioka section 3.4.2.

[7] In fact, once a nonzero number of extended states get partially populated, the longitudinal conductance R_{xx} becomes nonzero and hence we can no longer set $R_{xy} = \Sigma_{xy}^{-1}$.