Topological quantum computation: the general idea[1]

A very intriguing idea which has emerged over the last 10 years or so at the interface of traditional condensed matter physics and the younger field of quantum information is that one might be able to use the special properties of certain kinds of condensed matter systems, in particular the highly entangled nature of their groundstates, to carry out the operations necessary for quantum computation in a way which is *topologically protected*. Since the systems necessary for this program of topological quantum computation (hereafter TQC) are both intrinsically many-body and (at least in the most widely considered examples) intrinsically two-dimensional, this topic forms a natural capstone to this course. It should be emphasized right away that the idea of TQC is even more speculative than that of quantum computing in general, and as of now the experimental progress towards it has been minimal; nevertheless, the mere consideration of it throws up many intriguing challenges to our basic understanding of highly entangled condensed matter systems.

Let's start by briefly reviewing the requirements for a quantum computer in general.[2] In principle, what we need to construct a universal quantum computer, i.e. a quantum computer which is able to execute an arbitrary algorithm, is a set of quantum-mechanical systems confined to a two-dimensional Hilbert space ("qubits") and the ability to reliably perform on this collection of qubits a specific set of operations: initialization, readout and, most crucially, a set of unitary operations ("quantum gates") which must include *arbitrary* one-qubit gates (that is, unitary operations which change the state of a single qubit while leaving all the rest unchanged) and *at least one* two-qubit (entangling) gate. In the useful language of the "Bloch sphere", in which an arbitrary pure state of a single qubit can be specified by the direction of a unit "spin" $\boldsymbol{\sigma}$, we need to be able to perform arbitrary rotations of the different $\boldsymbol{\sigma}_i$ (where i labels the different qubits) and also at least one entangling two-qubit operation, e.g. the CNOT gate, which in terms of the Pauli matrices σ_ν ($\nu = x, y, z$) corresponds to the unitary operation

$$\hat{U}_{\text{CNOT}} = \frac{1}{2}\left\{(\hat{1} + \hat{\sigma}_z)_1\hat{1}_2 + (\hat{1} - \hat{\sigma}_z)_1\hat{\sigma}_{x2}\right\} \tag{1}$$

It is important to appreciate that even if we can reliably perform all these operations, it is necessary for our quantum computer to work that the state of the N-qubit system be reliably preserved (or evolve in an exactly known way) in the periods between the gate operations. The problem of designing a system which has this property, irrespective of whether or not we can perform all the gates necessary for universal quantum computing, is referred to as the problem of designing a "quantum memory". Thus, quantum memory

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[1]Some useful general references, in increasing order of "technicality": G. Collins, Sci. Am. **294**, 56 (2006); S. Das Sarma et al., Physics Today **59**, 32 (July 2006); J. Preskill, Lecture notes for Physics 219, Quantum Computation, http://www.theory.caltech.edu/~preskill/ph219; A. Stern, Ann. Phys. (NY) **323**, 204 (2008); C. Nayak et al., Rev. Mod. Phys. **80**, 1083 (2008).

[2]I will consider here only the standard paradigm ("circuit model") of quantum computing, as e.g. in the text of Nielsen and Chuang. In the context of TQC variant schemes such as "one-way" (cluster) and adiabatic quantum computing do not raise significantly different issues.

plus the set of one and two-particle[3] gates specified above is both a necessary and a sufficient condition for universal quantum computing; for a more restrictive set of tasks quantum memory plus a subset of the above gates may be adequate.

Of the many difficulties which impede the construction of a quantum computer in practice, two are particularly relevant in the context of TQC. First, it is often difficult to be sure that one has effected exactly the desired quantum gate. For example, if one's qubits are literal nuclear spins and one wishes, starting with the state $\sigma_z = +1$, to effect, say, a rotation through an angle θ around the x-axis, then an obvious way to do so is to impose an rf field in resonance with the dc field (assumed along $\hat{z}$); however, any error in either the length, the strength or the frequency of the pulse will lead to a final state different from the desired one, and in a quantum computer such errors, even if individually small, rapidly build up to the point where they may destroy the working of the computer.

The second major difficulty is decoherence: even in the "resting" periods between gates the effect of any unwanted and unplanned interaction of the qubits with the rest of the Universe (the "environment") will tend to destroy the delicate phase relations between the different branches of the superposition which are essential to the functioning of the quantum computer, or in other words it will destroy the "quantum memory". It is worth reminding ourselves that decoherence[4] can arise either from interaction with a part of the environment which is describable classically but only specified statistically (e.g., in the case of real nuclear spins, with the ambient $50\,\mathrm{Hz}$ background) or with a part which must itself be treated quantum-mechanically. In the first case the system remains, for any given realization of the classical field, in a pure state, but since we do not know which realization occurred and can only specify the probability of occurrence in a statistical way, the density matrix which we must use becomes a classical mixture: e.g. for a single qubit in a fluctuating magnetic field along the z-axis, the pure state DM

$$\hat{\rho}_{\text{pure}} = \begin{pmatrix} |a|^2 & a^*b \\ ab^* & |b|^2 \end{pmatrix} \tag{2}$$

is converted, through the interaction with our classical but statistically specified environment, into the mixture described by

$$\hat{\rho}_{\text{mixt}} = \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix} \tag{3}$$

In the second case (interaction with a quantum environment) the system and the environment become *entangled*. If χ denotes the state of the environment, then schematically, for an interaction diagonal in $\hat{\sigma}_z$

$$(a|\uparrow\rangle + b|\downarrow\rangle)|\chi_0\rangle \to a|\uparrow\rangle|\chi_\uparrow\rangle + b|\downarrow\rangle|\chi_\downarrow\rangle \tag{4}$$

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[3]But any completely entangling two-particle gate can substitute for CNOT.

[4]It is sometimes argued that the former case is not "real" decoherence. This seems to me a matter of linguistics, since the resulting density matrix is just as mixed as in the second case.

where $\chi_\uparrow$, $\chi_\downarrow$ are states of the environment which are in general different, and which for simplicity of illustration we will assume are in fact orthogonal. Since by hypothesis we cannot (or do not wish to) measure the state of the environment, we must trace over it, and thereby recover the transformation from $\hat{\rho}_{\text{pure}}$ (eqn. 2)) to $\hat{\rho}_{\text{mixt}}$ (eqn. (3)) just as in the case of a classical environment.

The above examples refer to a special case, where the coupling to the environment is only through the z-component of "spin" $\hat{\sigma}_z$. If the coupling also involves the transverse component of spin $\hat{\sigma}_x$ and/or $\hat{\sigma}_y$, then in addition to randomizing the relative phase of the amplitudes a and b and thereby effecting the transition from (2) to (3) ("phase flipping") it may also cause a change in the relative *magnitudes* of a and b, thus giving rise to a change in the expectation value $\langle\hat{\sigma}_z\rangle$ ("bit flipping"). In the literature it is common to describe the effects of decoherence phenomenologically by the Bloch equations of NMR theory: if we adopt the usual convention in which the $\hat{\sigma}_z$-representation corresponds to the eigenbasis of the Hamiltonian of the isolated system, these read

$$\frac{d\langle\sigma_z\rangle}{dt} = -\frac{\langle\sigma_z\rangle - \langle\sigma\rangle_{\text{eq}}}{T_1} \tag{5}$$

$$\frac{d\langle\boldsymbol{\sigma}_\perp\rangle}{dt} = -\frac{\langle\boldsymbol{\sigma}_\perp\rangle}{T_2} \tag{6}$$

Here $\langle\sigma_z\rangle_{\text{eq}}$ is the equilibrium value of $\langle\sigma_z\rangle$ under the relevant conditions, $\boldsymbol{\sigma}_\perp \equiv (\sigma_x, \sigma_y)$ is the transverse component of spin, and T_1 and T_2 are phenomenological relaxation times. Under rather general conditions one can write (since any field in the xy-plane tends to change one component of $\langle\boldsymbol{\sigma}_\perp\rangle$ as well as $\langle\sigma_z\rangle$)

$$\frac{1}{T_2} = \frac{1}{2T_1} + \frac{1}{T_\phi} \tag{7}$$

where T_ϕ is a "pure dephasing" time (corresponding to a fluctuating field along the z-axis). Thus, T_2 can never be longer than $2T_1$ and may be orders of magnitude shorter, as is the case in many physical systems which have been proposed as candidates for qubits. To build an ideal quantum memory (hence a fortiori and ideal quantum computer) one needs either that both T_1 and T_2 tend to infinity, or to be able to detect and correct the effects of decoherence when it occurs. The second option leads to the large topic of quantum error correction; in the present context we focus on the first option.

Why might topological considerations be relevant to the elimination of decoherence? And why may many-body systems be particularly useful in this context? To answer the first question, let's briefly consider the concept of "encirclement" of one physical object by another. As previously pointed out in the context of (abelian) anyons, this concept can only be unambiguously defined if the objects in question are restricted to move in a 2D physical (not Hilbert!) space. Under these conditions one can meaningfully ask the following question: Suppose an object starts at $\mathbf{r}_0$ at time zero, and some later time is again found at $\mathbf{r}_0$. In the intervening period, how many times did it encircle
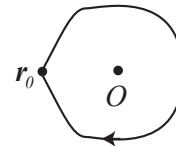
the point O (which may or may not be the position of a second physical object) and in what sense (clockwise or anticlockwise)? Suppose further that when O does represent the position of a second physical object (which may, but need not be, of identical type to the first) then the effect of (say) a clockwise encirclement is not just to multiply the wave function ψ_0 by a phase factor $\exp 2\pi i\alpha$ ($\alpha \neq 0, n$) as in the case of abelian anyons treated in lecture 18, but to *rotate it to a different state* ψ_1, and suppose furthermore that there is no way of telling ψ_0 and ψ_1 apart, by "local" interactions (i.e. those which only address the present position of the particle, not how it got there). Then it seems that ψ_0 and ψ_1 could form the basis of an ideal quantum memory; since no interaction with the environment can either tell them apart or mix them, the Hamiltonian of the system-environment interaction must be proportional to the unit matrix in the relevant 2D Hilbert space, so that a state of the form

$$\psi = \alpha|\psi_0\rangle + \beta|\psi_1\rangle \tag{8}$$

may be maintained indefinitely with 100% fidelity.

This scenario actually has a second advantage: the conversion from ψ_0 to ψ_1 was effected physically by the operation of "encirclement", and the latter is extremely robust; provided that the path we trace does not actually pass through O (something which is often excluded physically by the existence of, say, a large repulsive interaction) the the precise path followed does not matter; either 1 encircled 0 or 1 did not, and it doesn't matter how smooth or jagged the path was! Thus topological considerations may be able to guarantee not only total freedom from decoherence (an ideal quantum memory) but also the fidelity of quantum gate operations.

To get an idea why many-body systems may be particularly relevant, let's return briefly to the question of the metastability of supercurrents in superfluid ^{4}He, which was discussed in lecture 9. To a very crude approximation, one can think of the groundstate of superfluid ^{4}He in a toroidal geometry as a Bose condensate of atoms all in an s-state, while a metastable supercurrent in the same geometry corresponds to a Bose condensate of atoms all in a p-state. The p-state of a single atom can be distinguished from the s-state by its winding number

$$n \equiv \frac{1}{2\pi} \oint \boldsymbol{\nabla}\phi \cdot d\boldsymbol{l} \tag{9}$$

where ϕ is the phase of the Schrödinger wave function. Is n a "topological invariant"? For a *single* atom in the annular geometry, just as for an electron in an atom,[5] the answer is no; indeed, it is very easy to show that the ansatz

$$\psi = a(t)\psi_p + b(t)\psi_s \tag{10}$$

with $a(t)$ decreasing monotonically from 1 at $t = -\infty$ to 0 at $t = +\infty$, interpolates smoothly between the $n = 1$ and $n = 0$ states, and moreover, crucially, that on this

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[5]In the semiclassical approximation, in which the electromagnetic field is described classically and the electron can thus be described by a pure QM state. When the EM field has to be treated quantum-mechanically the situation is more complicated because of the atom-field entanglement.

trajectory the energy always decreases monotonically. For superfluid ^{4}He the situation is qualitatively different, since even with the simple "Gross-Pitaevskii" ansatz that all atoms remain always in a common state, the Fock term in the energy provides a large free energy barrier along the state described by eqn. (10). Thus for superfluid ^{4}He in an annular geometry the winding number is indeed a topological invariant.

Could one envisage the $n = 0$ and $n = 1$ states of superfluid ^{4}He as constituting a possible qubit? Alas, no: while the topological invariance of the winding number guarantees us that in the "obvious" basis where $\sigma_z = +1(-1)$ corresponds to $n = 1(0)$, the quantity T_1 is astronomically long, the quantity T_2 may be very short, since the "environment" (in this case principally the walls of the physical cell) can rather easily "tell the difference" between the s- and p-condensates. This suggests that we need to somehow combine the advantage of many-body physics and a strictly 2D physical space.

The systems actually envisaged for TQC are typically 2D many-body systems with groundstates that are usually degenerate and low-lying excitations which are *nonabelian anyons*, that is, crudely speaking, anyons such that encirclement operations lead not just to a phase factor (which happens for abelian anyons) but to an actual nontrivial rotation of the state vector in the relevant Hilbert space, so that successive encirclements, e.g. of 2 by 1 and 3 by 2, do not commute. Before going into the relevant theory in detail, let's first try to get some physical feeling for the meaning of abelian and nonabelian "statistics" and how they come to be realized.

To understand the way in which abelian fractional statistics may come about in a many-body system, it may be helpful to consider an analogous problem involving only two particles, which in turn can be understood by looking at two effects which can be illustrated in simple one-particle example.[6] The first effect, which we have already met, is "fractional charge" (or probability): to illustrate this notion we consider a single charged particle restricted to tunnel between the groundstate of two neighboring potential wells. Suppose we make a "projective measurement" of the charge Q_R to the right of the line $z = 0$ at some definite time t; then of course we will always find $Q_R = 1$ or 0. However, suppose we allow the system to relax to its groundstate, namely

$$\psi_0 = \cos\theta |R\rangle + \sin\theta |L\rangle, \quad \tan\theta \equiv \epsilon/\Delta \tag{11}$$

If we then perform a series of "weak" measurements over time so as to establish the *average* value of Q_R, then we will find a nonintegral result, $\langle Q_R \rangle = \cos^2\theta$ (which is of course what we would calculate). This does not seem particularly mysterious.

The second effect we need has to do with Berry's phase. Suppose that a QM system, say for definiteness a single particle, is subject to some classical control parameter λ (in general of the nature of a *vector* or something similar), so that its energy eigenfunctions, and in particular the GSWF, is a function of λ:

$$\Psi_0 = \Psi_0(\mathbf{r}, \lambda) \tag{12}$$

Suppose now that the parameter λ is varied *slowly* in time (i.e. over timescales long compared to $\hbar/E_{\min}$ where $E_{\min}$ is minimal excitation energy). If after some time T λ

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[6]The material of the next 3 pages (up to eqn. (25)) is repeated here for convenience from lecture 18. It will be reviewed only briefly in the lecture.

returns to its original value, then by our previous argument we must have

$$\Psi_{\text{fin}}(\mathbf{r}, \lambda) = \exp i\alpha \, \Psi_{\text{in}}(\mathbf{r}, \lambda) \tag{13}$$

Part of the phase increment α is a "dynamical" phase $\left(\hbar^{-1} \int_0^T E(\lambda(t)) \, dt\right)$ which depends on the actual time-dependence of $\lambda(t)$ (and on the arbitrary zero of energy). However, in general there may be a second contribution to α which is independent of the detailed dependence of $\lambda(t)$ and is a function *only of the path followed* (recall in the "interesting" case λ is of the nature of a vector or something similar). This is the celebrated *Berry phase*.

It is easy to obtain a formal expression for the Berry phase ϕ_{B} in terms of the variation of the groundstate[7] under a small adiabatic change $\delta\lambda$ of the control parameter λ. For such a change we evidently have (since Ψ_0 is normalized)

$$\delta(\arg \Psi_0) = \text{Im} \langle \Psi_0 | \delta\Psi_0 \rangle = \text{Im} \langle \Psi_0 | \partial\Psi_0/\partial\lambda \rangle \delta\lambda \tag{14}$$

and hence

$$\phi_{\text{B}} = \text{Im} \oint d\lambda \, \langle \Psi_0 | \partial\Psi_0/\partial\lambda \rangle \tag{15}$$

Note that it is irrelevant to the argument whether the expression on the RHS of (14) has a real part (although the corresponding contribution to (15) must vanish).

A very standard example of a nontrivial Berry phase is that of a spin-1/2 in magnetic field which is oriented in direction specified by θ, ϕ such that θ is constant in time but ϕ is rotated adiabatically (i.e. over a timescale $\ll (\mu B)^{-1}$ from 0 to 2π (see figure). For given θ, ϕ the groundstate satisfies

$$\boldsymbol{n} \cdot \hat{\boldsymbol{\sigma}} |\psi\rangle = |\psi\rangle \tag{16}$$

so if we require that the two component of the spinor wave function be single-valued as a function of ϕ, then the (almost) unique solution is, up to an overall complex but ϕ-independent constant,
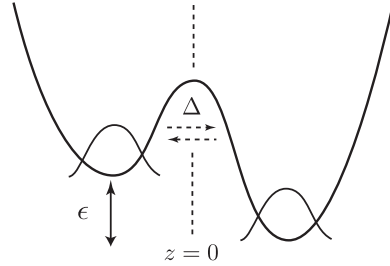
$$|\Psi\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \exp i\phi \end{pmatrix} \tag{17}$$
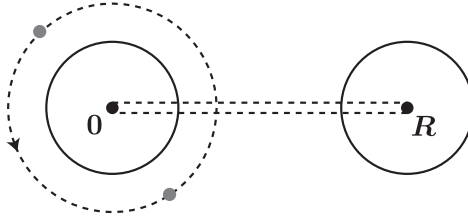
If we plug this form of wave function into formula (15) we find

$$\phi_{\text{B}} = \int_0^{2\pi} d\phi \, \sin^2\theta/2 = \pi(1 - \cos\theta) \tag{18}$$

where the RHS is the solid angle subtended by the "orbit" of the field; note that putting $\theta = \pi/2$ (i.e. rotating $\mathbf{B}$ through $360°$) gives the standard factor of -1 in the state of
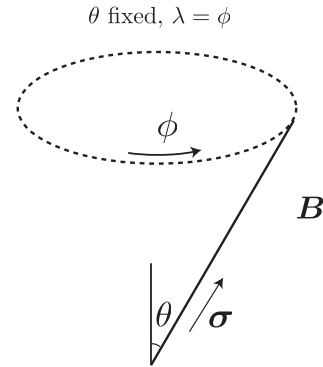
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[7]Or any other energy eigenstate.

a spin-1/2 particle. (Note that had we made the other possible "single-valued" choice, namely

$$|\Psi\rangle = \begin{pmatrix} \cos\theta/2 \exp -i\phi \\ \sin\theta/2 \end{pmatrix} \tag{19}$$

we would have got $\phi_{\mathrm{B}} = -\pi(1 + \cos\theta)$, which is equivalent to (18) modulo 2π).

Now let us put together the results of the two considerations above. As an example, imagine a quantum particle with a coordinate $\mathbf{r}_a$ which can be localized either at point $\mathbf{0}$ or at point $\boldsymbol{R}$ (or be in a quantum superposition of those two states). Further, imagine a second particle (with coordinate $\mathbf{r}_b$) which is constrained by energy considerations to be in a p-state relative to $\boldsymbol{a}$, i.e. the relative wave function must have the form (up to irrelevant dependence on $|\mathbf{r}_b - \mathbf{r}_a|$) $\exp i\phi$, where $\phi \equiv \arg(\mathbf{r}_b - \mathbf{r}_a)$. Then the most general wave function of the two-particle system will be of the form

$$\Psi(\mathbf{r}_a, \mathbf{r}_b) = a\Psi_{\mathbf{0}} \exp i\phi_{\mathbf{0}}(\mathbf{r}_b) + b\Psi_{\boldsymbol{R}} \exp i\phi_{\boldsymbol{R}}(\mathbf{r}_b) \tag{20}$$

where $\phi_{\mathbf{0}} \equiv \arg\mathbf{r}_b$, $\phi_{\boldsymbol{R}} \equiv \arg(\mathbf{r}_b - \boldsymbol{R})$. This wave function is of course perfectly single-valued with respect to $\mathbf{r}_b$. Now suppose we "pin" $\mathbf{r}_b$, e.g. with a stray external potential, and move it adiabatically around $\mathbf{0}$ as shown in the figure. ("Adiabatically" in this example means "on a timescale long compared to the inverse of the matrix element for the transition of $\mathbf{r}_a$ from $\mathbf{0}$ to $\boldsymbol{R}$). What is the resultant phase change (Berry phase) ϕ_{B}? We may parametrize the dashed path in the figure by $\lambda \equiv \phi_{\mathbf{0}}$; then we have

$$\frac{\partial\Psi}{\partial\lambda} = i\left\{a\Psi_{\mathbf{0}} \exp i\phi_{\mathbf{0}} + b\Psi_{\boldsymbol{R}} \frac{\partial\phi_{\boldsymbol{R}}}{\partial\phi_{\mathbf{0}}} \exp i\phi_{\boldsymbol{R}}\right\} \tag{21}$$

Since $\Psi_{\mathbf{0}}$ and $\Psi_{\boldsymbol{R}}$ are each assumed orthogonal and normalized, this means

$$\mathrm{Im}\,\langle\Psi|\partial\Psi/\partial\phi_{\mathbf{0}}\rangle = |a|^2 + |b|^2\partial\phi_{\boldsymbol{R}}/\partial\phi_{\mathbf{0}} \tag{22}$$

and the Berry phase ϕ_{B} is given by

$$\phi_{\mathrm{B}} = 2\pi|a|^2 + |b|^2 \int_0^{2\pi} (\partial\phi_{\boldsymbol{R}}/\partial\phi_{\mathbf{0}})\,d\phi_{\mathbf{0}} \tag{23}$$

However, since the path does not encircle $\boldsymbol{R}$, it is clear that the integral in (23), which is simply the total change in $\phi_{\boldsymbol{R}}$ due to the encirclement of $\boldsymbol{0}$, is zero. Hence

$$\phi_{\mathrm{B}} = 2\pi|a|^2 \neq 2\pi \tag{24}$$

or in words, since the mean charge enclosed by the path has the fractional value $|a|^2$,

$$\boxed{\text{statistical phase} = 2\pi \times \text{(fractional) average charge enclosed by path}} \tag{25}$$

(Thus, it is natural that the phase acquired by one Laughlin quasiparticle encircling another is $2\pi\nu$, and since this is *two* exchange processes, the exchange phase is $\pi\nu$.)

Nonabelian Anyons: Warmup

Example 1: Nonabelian Berry phases
Note that the expression

$$\varphi_B = i \int d\lambda \left\langle \Psi(\lambda) \Big| \frac{d\Psi}{d\lambda}(\lambda) \right\rangle$$

implicitly assumes GS $\Psi(\lambda)$ is non degenerate. But what if it is degenerate? Then in principle the quantity $\langle \Psi | d\Psi/d\lambda \rangle$ is a matrix:

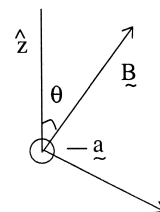$$K_{mn} \equiv \langle \Psi_m(\lambda) | d\Psi_n(\lambda) \rangle$$

Note that while K_{mn} must be antihermitian (to preserve orthonormality) its off-diagonal elements do not have to be pure imaginary.

A simple example: spin-1 particle with $\hat{H} = \alpha(\boldsymbol{S} \cdot \boldsymbol{B})^2, \alpha > 0$. GS is 2-fold degenerate: convenient to choose real eigenvectors in plane perpendicular to $\boldsymbol{B}$. Since these vectors must be single-valued as $f(\boldsymbol{B})$, a convenient choice is (with θ, φ polar and azimuthal angles of $\boldsymbol{B}$)

$\Psi_1 \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$
$\Psi_2 \equiv (\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}}) / |\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}}| = (-\sin\varphi, \cos\varphi, 0) \equiv \boldsymbol{a}$
$\Psi_3 \equiv (\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}})) / |\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}}| = (-\cos\theta\cos\varphi, -\sin\theta\sin\varphi, \sin\theta) \equiv \boldsymbol{b}$

So as $\boldsymbol{B}$ is carried adiabatically around circuit, $\boldsymbol{a}$ and $\boldsymbol{b}$ rotate through $\angle \chi$:

$\delta\chi = (\Psi_2, \partial\Psi_3/\partial\varphi) = \cos\theta_0$
$\Rightarrow \Delta\chi = \dfrac{\pi}{2}(1 - \cos\theta_0)$
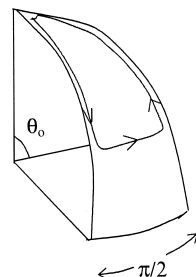
Example 2: The permutation group
Irreducible representations of the permutation group of N objects (S_N):

Permutations P_{ij} must satisfy $P_{ij}^2 = 1$ ($\forall \quad i, j$)

$N = 2$: trivial, only 2 generators, 1 and $\hat{P}_{12}$ (so abelian)
   2D Hilbert space,

$$\left. \begin{array}{ll} \Psi_s = |12\rangle + |21\rangle & \hat{P}_{12} = +1 \\ \Psi_a = |12\rangle - |21\rangle & \hat{P}_{12} = -1 \end{array} \right\} \text{abelian}$$

$N = 3 : 6$ "states", e.g. $|123\rangle, |213\rangle, ...$
6 indt. operators (e.g. $\hat{1}, \hat{P}_{12}, \hat{P}_{13}, \hat{P}_{23}, \hat{P}_{12}\hat{P}_{13}, \hat{P}_{13}\hat{P}_{12}$)

$$\Psi_s = |123\rangle + |132\rangle + |213\rangle + |231\rangle + |312\rangle + |321\rangle$$
$$\hat{P}_{12} = \hat{P}_{13} = \hat{P}_{23} = +1 \qquad \text{1-dim}^l.$$

$$\Psi_a = |123\rangle - |132\rangle - |213\rangle + |231\rangle + |312\rangle - |321\rangle$$
$$\hat{P}_{12} = \hat{P}_{13} = \hat{P}_{23} = -1 \qquad \text{1-dim}^l.$$

This leaves 4 states. Can choose to be 2 2-dim^l l reps., e.g.

$$\left. \begin{array}{ll} \Psi_s & = |123\rangle + \omega|231\rangle + \omega^2|312\rangle \\ \Psi_a & = |213\rangle + \omega|132\rangle + \omega^2|321\rangle \end{array} \right\} \omega = \exp 2\pi i/3$$

plus a second 2D rep with $\omega \to \omega^*$. In this basis

$$\hat{P}_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{P}_{13} = \begin{pmatrix} 0 & \omega^* \\ \omega & 0 \end{pmatrix}, \quad \hat{P}_{23} = \begin{pmatrix} 0 & \omega \\ \omega^* & 0 \end{pmatrix}$$

Note that in this basis the cyclic transformation $C(123)$ is diagonal:
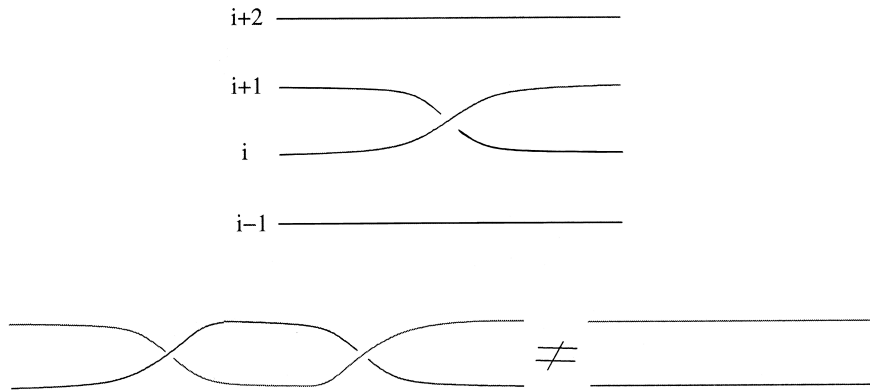
$$C(123) = \omega\hat{1}$$

Example 3: The Braid Group
The braid group B_n for n objects is constructed by numbering them arbitrarily $1, 2, ...n$
and defining
$$\hat{T}_i = \text{directed interchange of } i \text{ and } i + 1$$

($\hat{T}_n =$ directed interchange of n and 1). (e.g. "i over $i + 1$")
Note that $\hat{T}_i \neq \hat{P}_{i,i+1}$, and in particular we do not have $\hat{T}_i^2 = 1$:
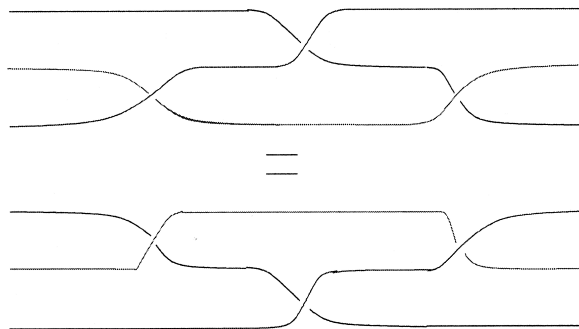
It is immediately obvious that the $\hat{T}_i$ satisfy

$$[\hat{T}_i, \hat{T}_j] = 0 \qquad \text{for } |i - j| > 1 \qquad (*)$$

Slightly less obviously,

$$\hat{T}_i \hat{T}_j \hat{T}_i = \hat{T}_j \hat{T}_i \hat{T}_j \qquad \text{for } |i - j| = 1 \qquad (**)$$

(*) and (**) may be taken as the definition of the braid group.

Rather than presenting the most general possible scheme for TQC,[8] it may be more useful in the present context to consider a specific implementation, based on a particular type of nonabelian excitations, the so-called Ising anyons (the name comes from the way these anyons appear in conformal field theory). As we shall see, the most promising (at present) physical implementations of TQC involve this type of excitation. Ising anyons have the following properties:

1. They can be created only in pairs (just as the abelian anyons of the Laughlin state can be created only in triples).

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[8]See e.g. Preskill, ref. cit.

2. The Hilbert space spanned by $2n$ anyons has dimension 2^n (not, as one might naively think, 2^{2n})

3. A qubit corresponds to a *pair* of anyons, but the allocation of anyon pairs between qubits is in some sense arbitrary. Thus $2n$ anyons give n qubits, which is consistent with the idea that the Hilbert space is 2^n-dimensional.

4. By various "encirclement" ("braiding") operations which involve carrying one anyon around another in the 2D physical space, it is possible to carry out the following operations (quantum gates) on the qubits:

$$\exp(i\pi\hat{\sigma}_z/4) \qquad \text{(effects e.g. } \tfrac{1}{\sqrt{2}}\binom{1}{1} \to \tfrac{1}{\sqrt{2}}\binom{1}{i} \\ \text{up to an overall phase)}$$

$$\exp(i\pi\hat{\sigma}_x/4) \qquad\qquad \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \tag{26}$$

$$\exp\tfrac{i\pi}{2}(\hat{\sigma}_{z1}\cdot\hat{\sigma}_{z2}) \quad \text{2-particle entangling gate}$$

and, needless to say, arbitrary product of such operations.

5. Apart from the anyons and the vacuum, which we shall denote by the symbol 1, there exist other excitations in the system and for a reason which will subsequently become apparent we will call them "fermions" - which, like the anyons, can actually be created (and absorbed) only in pairs.

6. The rules for production and recombination of the anyons and fermions ("fusion rules") are as follows:

    (a) Two fermions can recombine only to vacuum

    (b) A single anyon cannot recombine with a single fermion

    (c) Most importantly, two anyons can recombine in either of two ways: to the vacuum or to a single fermion

If we denote as above the vacuum by 1, a single fermion by ψ and a single anyon by σ (the notation is conventional in the field) we see that the rules (a)–(c) can be written formally as (cf. Stone and Chung, Phys. Rev. B **73**, 014505 (2006)):

$$\psi \times \psi = 1 \tag{27a}$$

$$\psi \times \sigma = \sigma \tag{27b}$$

$$\sigma \times \sigma = 1 + \psi \tag{27c}$$

The structure of a general nonabelian anyon theory can be regarded as defined by the combination of fusion rules exampled by (27) and the "braiding rules" which tell us how encirclement of one anyon by another executes various quantum gates.

It will be obvious from examination of point (4) above that while the operations which can be conducted in a topologically protected way using Ising anyons include a single 2-particle entangling gate as required, the set of (potential) single particle operations is very restricted, consisting of rotations of (multiples of) $\pi/4$ around the three orthogonal axes. (The set of such operations is sometimes called the "Clifford group" in the physics literature, although mathematicians appear to use this phrase in a much more general sense). Unfortunately, it is not possible to generate an arbitrary rotation on the Bloch sphere by using only these operations ("the Clifford group is not dense in $SU(2)$"), and thus Ising anyons as such do not permit completely topologically protected universal quantum computation; however, they do allow some more restricted quantum algorithms to be implemented in a completely topologically protected way, for example the Grover algorithm for $N = 4$ which turns out to require only the operations of the Clifford group plus a single 2-qubit entangling gate. If one wishes to guarantee that all possible single-qubit gates be topologically protected, one has to go to a different kind of anyons. The simplest alternative type is Fibonacci[9] anyons, which has a single type of anyonic object, denoted ϵ, and the single fusion rule

$$\epsilon \times \epsilon = 1 + \epsilon \tag{28}$$

Just as for Ising anyons, Fibonacci anyons permit the implementation of an entangling 2-qubit gate; however, in this case the permitted single-qubit operations are rotations of $\pi/5$ rather than $\pi/4$, around two orthogonal axes; this set of operations *is* "dense in $SU(2)$", i.e. by iterating it sufficiently we can generate an arbitrary rotation on the Bloch sphere. There is a possibility that a particular quantum Hall state (at $\nu = 12/5$) may be described by a groundstate wave function whose excitations included Fibonacci anyons.

To conclude this lecture, it is interesting to sketch the connection between the existence of anyons in a given physical system and the existence of degenerate and *mutually inaccessible groundstates*. For this purpose we consider a system which sustains *abelian* anyons, namely the QH state at $\nu = 1/3$. Evidently, in a simply connected geometry such as a disk the Laughlin ground state is nondegenerate. However, consider a Corbino-disk geometry, bend it round into a torus and impose (of course unphysically) the condition that the magnetic field be everywhere perpendicular to the surface and furthermore such that the filling factor ν ($\equiv$ number of electrons / flux quantum) is exactly 1/3. Then there is a remarkable theorem due to Wen and Niu:[10] the groundstate of such a system must inevitably be 3-fold degenerate, with the 3 states "mutually inaccessible" (this concept is defined below). It is very remarkable that the first part of the theorem *does not* require any specific knowledge of the groundstate wave function; the second part (regarding the mutual inaccessibility) does require the the excitations have a nonzero energy gap, but not that that the groundstate necessarily has the Laughlin

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[9]The name comes for the fact that the Hilbert space of n anyons has the dimension Fib(n), where Fib(j) is the j-th number of the Fibonacci series (1,2,3,5,8,13 ...).

[10]Phys. Rev. B **41**, 9377 (1990). Actually they prove a more general result, that for the QHE with $\nu = p/q$ (p, q mutually prime) on a surface of genus m, the degeneracy is q^m.

form (it would presumably apply, for example, to an unpolarized state with $\nu = 1/6$ per each spin, or to a Wigner crystal).

The proof of the theorem rests on the observation that in order to define the magnetic vector potential on a torus self-consistently, the area of the surface must correspond to an integral number of flux quanta. So imagine the torus "unfolded" to make a flat rectangular plate with sides L_1, L_2 and the magnetic field B normal to its plane; to mimic the toroidal setup we must then impose periodic boundary conditions both on the magnetic vector potential $\mathbf{A}$ (modulo a nonsingular gauge transformation) and on the electronic wave function ψ. The number of flux quanta is

$$N_s = L_1 L_2 / 2\pi l_{\mathrm{M}}^2 \tag{29}$$

where as usual $l_{\mathrm{M}} \equiv (\hbar/eB)^{1/2}$ is the magnetic length. Let the number of electrons be N_e, and such that

$$\frac{N_e}{N_s} \equiv \nu = 1/3 \tag{30}$$

For the moment let us neglect any potential energy, so that the Hamiltonian is just the KE term:

$$\hat{H} = \frac{1}{2}\left\{\left(-i\hbar\frac{\partial}{\partial x} + eA_x\right)^2 + \left(-i\hbar\frac{\partial}{\partial y} + eA_y\right)^2\right\} \equiv \hat{K} \tag{31}$$

Consider the pair of operators (usually called "magnetic translation operators") defined by

$$\hat{t}(\boldsymbol{a}) \equiv \exp i\hat{\mathbf{k}} \cdot \boldsymbol{a}/\hbar \tag{32a}$$

$$\hat{t}(\boldsymbol{b}) \equiv \exp i\hat{\mathbf{k}} \cdot \boldsymbol{b}/\hbar \tag{32b}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$ are vectors directed along orthogonal sides of the plate and the operators $\hat{k}_x$, $\hat{k}_y$ are defined by

$$\hat{k}_x \equiv -i\hbar\frac{\partial}{\partial x} + eA_x + eBy \tag{33a}$$

$$\hat{k}_y \equiv -i\hbar\frac{\partial}{\partial y} + eA_y - eBx \tag{33b}$$

(note the relative signs in the A and B terms, which are crucial!) It is easy to verify that $\hat{t}(\boldsymbol{a})$ and $\hat{t}(\boldsymbol{b})$ commute with $\hat{K}$ (eqn. (31)) and moreover commute with the boundary conditions (in the sense that if ψ satisfies those then so do $\hat{t}(\boldsymbol{a})\psi$ and $\hat{t}(\boldsymbol{b})\psi$, at least up to a nonsingular gauge transformation). However, $\hat{t}(\boldsymbol{a})$ and $\hat{t}(\boldsymbol{b})$ *do not commute with one another!* In fact, a simple calculation shows that

$$\hat{t}(\boldsymbol{a})\hat{t}(\boldsymbol{b}) = \hat{t}(\boldsymbol{b})\hat{t}(\boldsymbol{a}) \exp -i(\boldsymbol{a} \times \boldsymbol{b})/l_{\mathrm{M}}^2 \tag{34}$$

Consider now the many-body problem defined by the Hamiltonian

$$\hat{H} = \sum_i \hat{K}_i + \frac{1}{2}\sum_{ij} \hat{V}(\mathbf{r}_i - \mathbf{r}_j) \tag{35}$$

(where we have still for the moment set the *single*-particle potential $V(\mathbf{r}_i)$ equal to zero).
Define the operator

$$\hat{T}(\boldsymbol{a}) \equiv \prod_{i=1}^{N_e} \hat{t}_i(\boldsymbol{a}) \tag{36}$$

i.e. we perform a magnetic translation of all the N_e electrons through $\boldsymbol{a}$. It is clear that
$\hat{T}(\boldsymbol{a})$ commutes with $\hat{V}$ as well as with $\hat{K}$. Consider the special choices

$$\hat{T}_1 \equiv \hat{T}(L_1 \hat{\boldsymbol{x}}/N_s), \quad \hat{T}_2 \equiv \hat{T}(L_2 \hat{\boldsymbol{y}}/N_s) \tag{37}$$

where $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are unit vectors along the two orthogonal sides of the plate. It is essential
to note that the arguments of $\hat{T}_1$ and $\hat{T}_2$ are *very small*:

$$|L_1 \hat{\boldsymbol{x}}/N_s| = L_1/(L_1 L_2 l_{\mathrm{M}}^{-2}) = l_{\mathrm{M}}^2/L_2 \ll l_{\mathrm{M}}, \quad \text{etc.} \tag{38}$$

Now comes the crunch: We have the relations

$$[\hat{T}_1, \hat{H}] = [\hat{T}_2, \hat{H}] = 0 \tag{39}$$

(and $\hat{T}_1, \hat{T}_2$ leave the boundary condition invariant), but

$$\hat{T}_1 \hat{T}_2 = \hat{T}_2 \hat{T}_1 \exp -2\pi i/3 \tag{40}$$

In view of (39), we may choose (a particular) groundstate ψ_0 to be an eigenstate of (e.g.)
$\hat{T}_2$. Since $\hat{T}_2$ is unitary, this means we must have

$$\hat{T}_2 \psi_0 = (\exp i\lambda) \psi_0 \tag{41}$$

But then, applying (40) to (41), we find that $\hat{T}_1 \psi_0$ is also an eigenstate of $\hat{T}_2$, with a
different eigenvalue $\exp i(\lambda + 2\pi/3)$, and $\hat{T}_1^2 \psi_0$ similarly with eigenvalue $\exp i(\lambda + 4\pi/3)$.
However, $\hat{T}_1^3 \psi_0$ has the same eigenvalue of $\hat{T}_2$ as ψ_0 and therefore is (or at least may
be) the same state. Thus, at least $\hat{T}_1 \psi_0$ and $\hat{T}_1^2 \psi_0$ must be *different states*[11] from ψ_0
(and from one another). On the other hand, in view of eqn. (39) they are eigenstates
of $\hat{H}_0$ with the same eigenvalue as ψ_0! Hence the groundstate must be (at least) 3-fold
degenerate.

   The argument is not yet complete, because we have still to take into account the
effect of the single-particle potential $\hat{U} \equiv \sum_i \hat{U}(\mathbf{r}_i)$. Indeed, the quantities $\hat{t}(\boldsymbol{a})$ and $\hat{t}(\boldsymbol{b})$
do not in general commute with $\hat{U}$, so the argument as given above fails. Nevertheless,
Wen and Niu show that in view of the smallness of the arguments of $\hat{T}_1$ and $\hat{T}_2$ relative
to the characteristic scale of variation of the potential (which in real life is presumably
at most of the order of the atomic size), the matrix elements of $\hat{U}$ in the groundstate
manifold are simply proportional to the unit matrix plus a term of the order of $\exp -L/\xi$
where ξ is some microscopic length. I give here a "poor man's version" of their argument:

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[11] Note that were they same states up to a phase factor they would necessarily have the same eigenvalue
of $\hat{T}_2$.

Since the operators $\hat{T}_1^3$ and $\hat{T}_2^3$ commute, we may take (any one of) the ground-state(s) to be a simultaneous eigenstate of these two, with eigenvalue $\exp 2\pi i$ in each case (cf. above). Since $\hat{T}_0(\boldsymbol{a})$ can be written as $\exp i\boldsymbol{Q} \cdot \boldsymbol{a}$ where $\boldsymbol{Q} \equiv \sum_i \mathbf{k}_i$ is the total "magnetic pseudomomentum" this strongly suggests (though it does not prove) that the state is strongly oscillating along the x-axis as a function of the COM coordinate $\mathbf{R}$, with a wavelength of the order of the sub-microscopic length $l_{\text{osc}} \sim L_1/N_s \sim l_{\text{M}}^2/L_2$. The effect of going from ψ_0 to $\hat{T}_1\psi_0$ is then likely to be qualitatively similar to shifting the COM through l_{osc}. Thus, we may estimate the off-diagonal elements of $\hat{U}$ in the groundstate manifold as proportional to the Fourier transform $U(\boldsymbol{Q})$ with respect to the COM coordinate $\mathbf{R}$, with $\boldsymbol{Q} \sim L_2/l_{\text{M}}^2$, that is

$$\hat{U}_{o-d} \equiv \langle \hat{T}_1\psi_0|\hat{U}|\psi_0\rangle \sim$$
$$\int \psi_0(\mathbf{R}) \exp i\boldsymbol{Q} \cdot \mathbf{R}\, U(\mathbf{R}, \ldots)\psi_0(\mathbf{R})\, d\mathbf{R} \sim \int U(\mathbf{R})\, \exp i\boldsymbol{Q} \cdot \mathbf{R}\, d\mathbf{R} \tag{42}$$

(since $\psi_0(\mathbf{R})$ is appropriately normalized). But barring pathology,[12] the *minimum* distance over which $U(\mathbf{R})$ changes appreciably (i.e. by a factor ~ 1 relative to its "typical" value) is $\sim$ the length l_U over which the *single-particle* potential $U(\mathbf{r})$ changes, which in turn is at least of the order of the atomic size. Using standard results on the Fourier transforms of slowly varying functions we thus conclude that

$$\hat{U}_{o-d} \lesssim \mathcal{O}(\exp - l_U L_2/l_{\text{M}}^2) \sim \exp - L_2/\xi \tag{43}$$

where the length $\xi \equiv l_{\text{M}}^2/l_U$, though possibly a couple of orders of magnitude larger than l_{M}, is still "microscopic", so that $\hat{U}_{0-d}$ vanishes exponentially with the sample size. A similar argument shows that the difference between the diagonal elements is of the same order of magnitude, so within the groundstate manifold

$$\hat{U}_{o-d} = U_0\hat{1} + o(e^{-L/\xi}), \quad \text{QED} \tag{44}$$

What does the triply degenerate groundstate have to do with the fractional $(1/3)$ statistics of the anyons of the $\nu = 1/3$ FQHE state? I *think* the answer is that while in a simply connected geometry the effect of "encircling" one anyon with another simply introduces a phase factor, in a toroidal geometry it may, depending on whether the "encirclement" involves topologically nontrivial motion around the torus, bring us back to a different many-body state. Then if one starts from a particular groundstate, say ψ_0, creates 3 anyons, performs some topologically nontrivial encirclement operation and then annihilates the anyons, one may return to a *different* groundstate. (This question has almost certainly been discussed for the FQHE somewhere in the literature...)

In the next lecture we will see that we can in fact make some precise statements about the relationship between groundstate degeneracy and anyonic statistics in the case of a specific exactly soluble model, the Kitaev "toric code".

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[12]It needs to be investigated whether formation of a crystal lattice counts as "pathology".