$p + ip$ Fermi superfluids[1]

As mentioned in lecture 25, a leading candidate for (Ising) topological quantum computation is a degenerate 2D Fermi system that forms Cooper pairs in the so-called $p + ip$ state. In this case the anyons are constituted by *vortices*; these vortices may or may not carry "Majorana fermions," which as we shall see are essentially the two halves of a "split" Dirac fermion, so that a single Dirac fermion is shared by two vortices; a qubit is essentially formed by a *pair* of vortices, so that the Hilbert space corresponding to a $2n$ vortices is 2^n-dimensional. The vortices are believed to be the analogs of the fractionally charged quasiparticles of the Moore-Read state, which possibly describes the $\nu = 5/2$ QHE, and it is believed that by braiding them appropriately one can implement nonabelian (Ising) statistics. Candidate systems for a 2D $p + ip$ Fermi superfluid include p-wave-paired Fermi alkali gases, with either one or more than one hyperfine species (a system yet to be realized experimentally) and, among existing systems, the superfluid A phase of liquid ^{3}He confined to a thin slab and, most importantly, strontium ruthenate (Sr_2RuO_4)[2]; both these systems contain 2 spin species, which to a first approximation may be regarded as forming Cooper pairs independently (though see below). For simplicity I start by considering the so far unrealized single species ("spinless") case, and return later to the generalization of the argument to the more realistic 2-species case. I will first give the "orthodox" account[3], and subsequently raise some questions about it.

Strontium Ruthenate (Sr_2RuO_4)[4]

Strongly layered structure, anal. cuprates ⇒ hopefully sufficiently "2D." Superconducting with $T_c \sim 1.5$ K, good type-II props. (⇒ "ordinary" vortices certainly exist).

$64 K question: is pairing spin triplet ($p + ip$)?

Much evidence both for spin triplet and for odd parity ("p not s").

Evidence for broken T-reversal symmetry:

optical rotation (Xia et al. (Stanford), 2006)

Josephson noise (Kidwingira et al. (UIUC), 2006)

Can we generate HQV's in SR_2RuO_4?

Problem:

in neutral system, both ordinary and HQ vortices have $1/r$ flow at ∞ ⇒ HQV's not specially disadvantaged. In charged system (metallic superconductor), ordinary vortices

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[1] Some material in these notes (marked by arrows in the margin) is reproduced for convenience from lecture 25, and will be discussed only briefly in the lecture

[2] Not to be confused with $Sr_3Ru_2O_7$, which is a very interesting system in its own right but does not form Cooper pairs.

[3] The seminal papers are Read and Green, Phys. Rev. B **61**, 10267 (2000) and D. A. Ivanov, PRL**86**, 268 (2001).

[4] Mackenzie and Maeno, Rev. Mod. Phys. **75**, 688 (2003)

have flow completely screened out for $r \gtrsim \lambda_L$ (London penetration depth) by Meissner effect (fig.1a). For HQV's, this is not true (fig 1b):

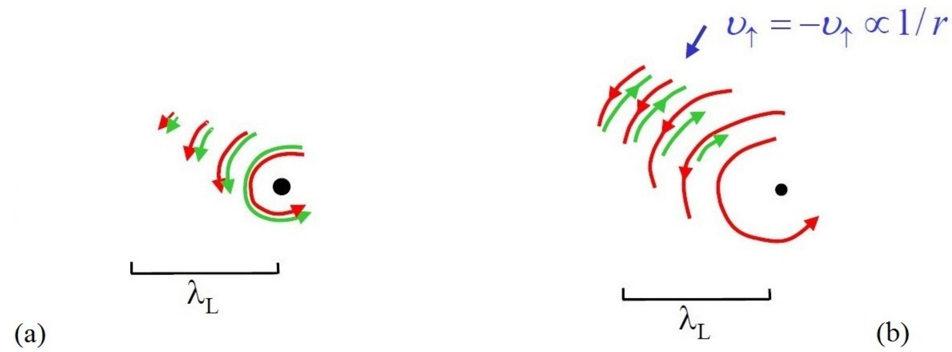


Figure 1:

So HQV's intrinsically disadvantaged in Sr_2RuO_4. However, Jang et al. (UIUC, 2011: see below) produce strong evidence for at least *single* HQV's.

(Note on $Cu_xBi_2Se_3$)

The orthodox account

The generic "particle-conserving" BCS ansatz for N spinless fermions (N even) is ←↓

$$\Psi_N = \left(\sum_{\mathbf{k}} c_{\mathbf{k}} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} \right)^{N/2} |\text{vac}\rangle, \text{ where } c_{\mathbf{k}} = -c_{-\mathbf{k}} \text{ (from antisymmetry)} \tag{1}$$

In the literature, it is more common to use the PNC (particle non-conserving) form:

$$\Psi_{\text{BCS}} = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}) |\text{vac}\rangle, \; u_{\mathbf{k}} = u_{-\mathbf{k}}, \; v_{\mathbf{k}} = -v_{-\mathbf{k}} \tag{2}$$

with

$$u_{\mathbf{k}} \equiv \frac{1}{(1 + |c_{\mathbf{k}}|^2)^{1/2}}, \; v_{\mathbf{k}} \equiv \frac{c_{\mathbf{k}}}{(1 + |c_{\mathbf{k}}|^2)^{1/2}} \tag{3}$$

so that $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 \equiv 1$, $c_{\mathbf{k}} = v_{\mathbf{k}}/u_{\mathbf{k}}$.
Two important quantities in BCS theory are

$$\langle n_{\mathbf{k}} \rangle = |v_{\mathbf{k}}|^2, \; F_{\mathbf{k}} \equiv \langle a_{-\mathbf{k}} a_{\mathbf{k}} \rangle_{\text{BCS}} = u^*_{\mathbf{k}} v_{\mathbf{k}} = \frac{c_{\mathbf{k}}}{1 + |c_{\mathbf{k}}|^2} \tag{4}$$

The Fourier transform of $F_{\mathbf{k}}$, $F(\mathbf{r})$ ($\equiv \langle \hat{\psi}(0) \hat{\psi}(\mathbf{r}) \rangle_{\text{BCS}}$) plays the role of the wave function of the Cooper pairs.

In standard BCS ("mean-field") theory, one minimizes the sum of the kinetic energy $\langle T\rangle = \sum_{\mathbf{k}}(\epsilon_{\mathbf{k}} - \mu)\langle n_{\mathbf{k}}\rangle$ and the "pairing" part of the potential energy

$$\langle V_{\text{pair}}\rangle = \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\langle a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}'} a_{\mathbf{k}'}\rangle \qquad \big(V_{\mathbf{k}\mathbf{k}'} \equiv \langle \mathbf{k},\ -\mathbf{k}|V|\mathbf{k}',\ -\mathbf{k}'\rangle\big) \tag{5}$$

Then the pair wavefunction $F_{\mathbf{k}}$ satisfies the Schrödinger-like equation:

$$2E_{\mathbf{k}}F_{\mathbf{k}} = -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}F_{\mathbf{k}'} \tag{6}$$

with

$$E_{\mathbf{k}} \equiv \frac{|\epsilon_{\mathbf{k}} - \mu|}{(1 - 4|F_{\mathbf{k}}|^2)^{1/2}} \equiv E_{\mathbf{k}}[F_{\mathbf{k}}] \tag{7}$$

Eqn. (6) is a disguised form of the BCS gap equation:

$$\begin{aligned} \Delta_{\mathbf{k}} &= -\textstyle\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\Delta_{\mathbf{k}'}/2E_{\mathbf{k}'} \\ E_{\mathbf{k}} &= \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + |\Delta_{\mathbf{k}}|^2} \end{aligned} \tag{8}$$

Note that the gap equation refers to the Cooper pairs (condensate). However, in the spatially uniform case $E_k \equiv \sqrt{\epsilon_k - \mu)^2 + |\Delta_k|^2}$ also represents the energy of excitation of single quasiparticle of momentum k: in the PNC formalism

$$\Psi_0 = \prod_{\mathbf{k}>0} \big(u_{\mathbf{k}}|00\rangle_{\mathbf{k}} + v_{\mathbf{k}}|11\rangle_{\mathbf{k}}\big) \equiv \prod_{\mathbf{k}>0} \Phi_{\mathbf{k}}^{(0)} \tag{9}$$

$$\Psi^{(\mathbf{k}_0)} = \prod_{\mathbf{k}\neq\mathbf{k}_0} \Phi_{\mathbf{k}}^{(0)} \cdot |01\rangle_{\mathbf{k}_0} \qquad (or\ \ \ldots|10\rangle_{\mathbf{k}_0}) \tag{10}$$

where $|01\rangle_{\mathbf{k}}$ means the state with $\mathbf{k}$ empty and $-\mathbf{k}$ occupied, etc.

In 2D, a possible p-wave solution of gap equation is

$$F_{\mathbf{k}} = (k_x + ik_y)f(|\mathbf{k}|)\big(\equiv (p_x + ip_y)f(|\mathbf{p}|),\ \text{hence "}p+ip\text{"}\ \big) \tag{11}$$

then also $\Delta_{\mathbf{k}} = (k_x + ik_y)g(|\mathbf{k}|)$, $E_{\mathbf{k}} = h(|\mathbf{k}|)$ ($\hat{\mathbf{k}}-$indep.) $\neq 0$, $\forall\mathbf{k}$.

It is important to note that the energetics is determined principally by the form of $F_{\mathbf{k}}$ close to Fermi energy ($|\epsilon_{\mathbf{k}} - \mu| \lesssim \Delta_0 \leftarrow\equiv |\Delta_{\mathbf{k}}|_{\mathbf{k}=\mathbf{k}_F}$). But for TQC applications, we may need to know $F_{\mathbf{k}}$ very far from the Fermi surface ($k \to 0$ and/or $k \to \infty$). Note that in most real-life cases (but possibly not in the yet-to-be-obtained ultracold Fermi gas case) we have

$$\Delta_0 \ll \mu \qquad \text{("BCS limit")} \tag{12}$$

Some properties of the $(p + ip)$ state: $\leftarrow\uparrow$

(a) Angular momentum:
Recall: $\Psi_N = \left(\sum_{\mathbf{k}} c_{\mathbf{k}} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}\right)^{N/2} |\text{vac}\rangle \equiv \hat{\Omega}^{\mathrm{N}/2}|\text{vac}\rangle$
with $c_{\mathbf{k}} = v_{\mathbf{k}}/u_{\mathbf{k}} \equiv u^*_{\mathbf{k}} v_{\mathbf{k}}/|u_{\mathbf{k}}|^2 = F_{\mathbf{k}}/(1-\langle n_{\mathbf{k}}\rangle) \propto \exp i\varphi_{\mathbf{k}}$
Since $\hat{L}_z = -i\hbar\partial/\partial\varphi$, $\left[\hat{L}_z,\ \hat{\Omega}\right] = \hbar$ and so (since $\hat{L}_z|\text{vac}\rangle \equiv 0$)

$$\hat{L}_z\Psi_N = \frac{N\hbar}{2}\Psi_N \tag{13}$$

leading to a macroscopic discontinuity at point $|\Delta_0| \to 0$. More seriously, at first sight as $T \to T_c$ from below $\langle L_z\rangle \sim \frac{N\hbar}{2}\big(1 - \mathcal{O}(T_c/\epsilon_{\mathrm{F}})\big)$! (cf. however T. Kita, JPSJ 67, 216 (1998))

(b) Real-space MBWF in long-distance limit:
In 1^{st}-quantized, real-space representation,

$$\Psi_N \equiv \Psi_N\{z_i\} = \mathrm{Pf}\,[f(z_i - z_j)] \tag{14}$$

where $z_i \equiv x_i + iy_i$ and $f(z)$ is the FT of $c_{\mathbf{k}}$.
At long distances $|z_i - z_j|$, $f(z_i - z_j)$ should be determined by the $k \to 0$ behavior of $c_{\mathbf{k}}$:

$$\begin{aligned} c_{\mathbf{k}} = F_{\mathbf{k}}/|u_{\mathbf{k}}|^2 = (\Delta_{\mathbf{k}}/E_{\mathbf{k}})/\left(1 + \tfrac{(\epsilon_{\mathbf{k}}-\mu)}{E_{\mathbf{k}}}\right) \\ \left(E_{\mathbf{k}} \equiv +\sqrt{(\epsilon_{\mathbf{k}}-\mu)^2 + |\Delta_{\mathbf{k}}|^2}\right) \end{aligned} \tag{15}$$

For a $(p+ip)$ state, $\Delta_{\mathbf{k}} \propto (k_x + ik_y)g(|\mathbf{k}|)$, so unless $g(0) = 0$, we find[5] as $\mathbf{k} \to 0$: $\Delta_{\mathbf{k}} \to \text{const.}(k_x + ik_y)$, $\left(1 - \frac{|\epsilon_{\mathbf{k}}-\mu|}{E_{\mathbf{k}}}\right) \to 2|\Delta_{\mathbf{k}}|^2/\mu$, so $c_{\mathbf{k}} \to \text{const.}/(k_x - ik_y)$ so the FT $F(z_i - z_j)$ behaves at large distances as $(z_i - z_j)^{-1}$; this then implies

$$\Psi_N \sim \mathrm{Pf}\left\{\frac{1}{z_i - z_j}\right\} \tag{16}$$

Note: this conclusion depends on behavior of $\Delta_{\mathbf{k}}$ (etc.) very far from F.S.

Bogoliubov-de Gennes (BdG) equations ↙

In the simple spatially uniform case, a simple relation exists between the "completely paired" state of $2N$ particles and the $(2N+1)$-particle states ("quasiparticle excitations")– the BCS wavefunction is product of states $(\mathbf{k}, -\mathbf{k})$, the excitations involve breaking single pair as in eqn. (10). In the general case no such simple relationship exists: nevertheless, BdG equations enable us to relate $(2N+1)$-particle states to $(2N)$-particle GS. (They do

[5] *Provided* that $\mu > 0$. For $\mu > 0$ (the so-called "trivial" phase) $c_k \to$ const. as $\mathbf{k} \to 0$.

not tell us directly about the $(2N)$-particle GS itself). The standard (PNC) approach goes as follows: The exact Hamiltonian is

$$\hat{H} - \mu\hat{N} = \int d\mathbf{r}\psi^\dagger(\mathbf{r})\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) - \mu\right) + \iint d\mathbf{r}d\mathbf{r}'\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}) \tag{17}$$

where $U(\mathbf{r})$ is the single-particle potential. In PE term, make mean-field approximation:

$$\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}) \to \Delta(\mathbf{r},\ \mathbf{r}')\psi^\dagger(\mathbf{r}')\psi^\dagger(\mathbf{r}) + H.C. \tag{18}$$

where

$$\Delta(\mathbf{r},\ \mathbf{r}') \equiv \int V(\mathbf{r}-\mathbf{r}')\langle\psi(\mathbf{r}')\psi(\mathbf{r})\rangle \quad (= \text{c-number}) \tag{19}$$

So:

$$\hat{H} - \mu\hat{N} = \int d\mathbf{r}\hat{\psi}^\dagger(\mathbf{r})\hat{H}_0\hat{\psi}(\mathbf{r}) + \left\{\iint d\mathbf{r}d\mathbf{r}'\Delta(\mathbf{r},\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}') + \text{H.c.}\right\} \tag{20}$$

which is a bilinear form and can be diagonalized

In this (PNC) formalism, the GS is a superposition of even-N states. Similarly, the excitations are superpositions of odd-N states and are generated by operators of the form (operating on the GS)

$$\gamma_n^\dagger = \int d\mathbf{r}\left\{u_n(\mathbf{r})\psi^\dagger(\mathbf{r}) + v_n(\mathbf{r})\psi(\mathbf{r})\right\} \tag{21}$$

with (positive) energies E_n (so $\hat{H} - \mu\hat{N} = \sum_n E_n\gamma_n^\dagger\gamma_n + \text{const.}$)

To obtain the eigenvalues E_n and eigenfunctions $u_n(\mathbf{r})$, $v_n(\mathbf{r})$ of the MF Hamiltonian, we need to solve the equation

$$[\hat{H} - \mu\hat{N}, \gamma_n^\dagger] = E_n\gamma_n^\dagger \tag{22}$$

Explicitly, this gives the BdG equations[6]

$$\hat{H}_0 u_n(\mathbf{r}) + \int \Delta(\mathbf{r},\mathbf{r}')v_n(\mathbf{r}')d\mathbf{r}' = E_n u_n(\mathbf{r}) \tag{23a}$$

$$\int \Delta^*(\mathbf{r},\mathbf{r}')u_n(\mathbf{r}')d\mathbf{r}' - \hat{H}_0^* v_n(\mathbf{r}) = E_n v_n(\mathbf{r}) \tag{23b}$$

$$\left(\hat{H}_0 \equiv -\tfrac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) - \mu\right)$$

General properties of solutions of BdG equations:

[6] Note that in absence of magnetic vector potential, $\hat{H}_0^* = \hat{H}_0$.

1. For $E_n \neq E_{n'}$, the spinors $\binom{u_n(\mathbf{r})}{v_n(\mathbf{r})}$ are mutually orthogonal, i.e., we can take

$$(u_n,\ u_{n'}) + (v_n,\ v_{n'}) = \delta_{nn'} \qquad \left((f,\ g) \equiv \int f^*(\mathbf{r})g(\mathbf{r})\,d\mathbf{r}\right) \tag{24}$$

2. If $\binom{u}{v}$ is a solution with energy E_n, then $\binom{v^*}{-u^*}$ is a solution with energy $-E_n$. For $E_n \neq 0$ the negative-energy solutions are conventionally taken to describe the "filled Fermi sea."

3. Under special circumstances, it may be possible to find a solution corresponding to $E_n = 0$ *and* $u_n(\mathbf{r}) = v_n^*(\mathbf{r})$. In this case

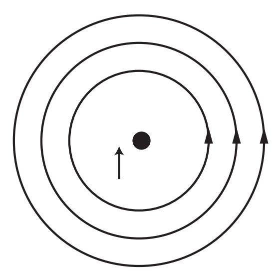
$$\begin{aligned}\hat{\gamma}_n &\equiv \int d\mathbf{r}\{u_n^*(\mathbf{r})\hat{\psi}(\mathbf{r}) + v_n^*(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\} \\ &= \int d\mathbf{r}\{v_n(\mathbf{r})\hat{\psi}(\mathbf{r}) + u_n(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\} \equiv \hat{\gamma}_n^\dagger\end{aligned} \tag{25}$$

i.e., the "particle" is its own antiparticle! Such a situation is said to describe a *Majorana fermion* (MF). (Note: this can only happen when the paired fermions have parallel spin, otherwise particle and antiparticle would differ by their spin) ←↑

Vortex in an s-wave Fermi superfluid

In a homogeneous s-wave superconductor, the gap $\Delta_{\mathbf{k}}$ is not appreciably a function of the relative mom. $\mathbf{k}$ of electrons in a Cooper pair in the region near k_{F}. So, when we consider an inhomogeneous situation, we can write Δ simply as a function $\Delta(\mathbf{R})$ of the COM coordinate $\mathbf{R}$ of the pairs; the form of $\Delta(\mathbf{R})$ must eventually be determined self-consistently. Note that $\Delta(\mathbf{R})$ is, apart from a constant factor, the quantity

$$F(\mathbf{R}) \equiv \langle \psi_\uparrow^\dagger(\mathbf{R})\psi_\downarrow^\dagger(\mathbf{R})\rangle \tag{26}$$



so it is a 2-particle quantity.

A vortex in an s-wave superconductor is described by a $\Delta(\mathbf{R})$ of the form (for all $R \gg \xi$, where ξ is the pair radius).

$$\Delta(\mathbf{R}) = f(|\mathbf{R}|)\exp i\varphi \tag{27}$$

$$(f(|\mathbf{R}|) \to 0 \text{ for } R \to 0)$$

Such a vortex has a circulation (at $r \ll \lambda_L$) of $h/2m$. Note that at first sight the form (27) violates the SVBC (single-valuedness boundary condition): this is usually hand-waved away by noting[7] that the form (27) needs to be modified for $r \lesssim \xi$.

In the neutral case, the (mass) current is simply proportional to $\boldsymbol{\nabla}(\arg\Delta(\mathbf{R}))$, so is of the form $\hat{\boldsymbol{z}} \times \mathbf{R}/R^2$ out to arbitrary distances. Thus, the "quantum of circulation" $\kappa \equiv \oint \boldsymbol{v}_s \cdot d\boldsymbol{l} = h/2m$.

[7] For a careful discussion of a closely related point see V. Vakaryuk, PRL **101**, 167002 (2008).

$p + ip$ Fermi superfluid (F.S.)
Spinless case (↑↑ only, say) (Fermi alkali gases)

The orbital wf $F(\mathbf{r},\ \mathbf{R}) \equiv \langle \hat{\psi}_\uparrow(\mathbf{R} + \mathbf{r}/2)\hat{\psi}_\uparrow(\mathbf{R} - \mathbf{r}/2)\rangle$, so cannot be written as a function of the COM variable $\mathbf{R}$ alone; thus neither can the "gap" Δ. In homogeneous bulk (F independent of the COM coordinate) various dependences on $\mathbf{r}$ are possible: the "$p + ip$" state is defined by having

$$F(|\mathbf{r}|) = (x + iy)F(|\mathbf{r}|) \tag{28}$$

or in momentum space, near Fermi surface,

$$F(\mathbf{p}) = (p_x + ip_y) \qquad \text{(hence name)} \tag{29}$$

In a BCS-like theory in 2D, it is the energetically favored state. In principle the "gap" Δ should be written as a function of both $\mathbf{R}$ and $\mathbf{r}$. In practice it is usually written as

$$\Delta = p_{\mathrm{F}}^{-1}\Delta_0(\mathbf{R})(\nabla_x + i\nabla_y)\delta(\mathbf{r}) \tag{30}$$

(where the p_{F} is inserted so that $\Delta_0(\mathbf{R})$ has the dimensions of energy), with the understanding that the ∇ acts on the relative coordinate.

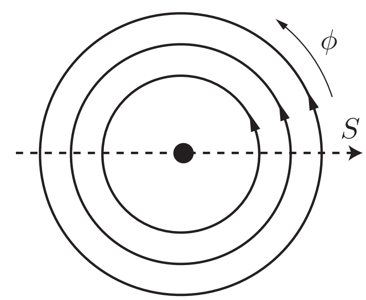
Vortices in a spinless ($p + ip$) F.S. neutral case:

The structure is similar to that in s-wave (BCS) case, with two differences; in that case vortices with $\Delta \propto e^{i\Phi}$ and "antivortices" with $\Delta \propto e^{-i\Phi}$ were equivalent by symmetry, in the $(p + ip)$ case we cannot assume this a priori.

Second difference with BCS: *we expect Majorana anyons.*

Existence of Majorana mode

Semiclassical approach[8]:



$$\hat{H}_{\mathrm{BdG}} = \begin{pmatrix} \hat{H}_0 & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -\hat{H}_0 \end{pmatrix}$$
$$\hat{H}_0 \equiv -\frac{\hbar^2}{2m}\nabla^2 - \mu \tag{31}$$

$\Delta(\mathbf{r})$ is approximated by $\Delta(\mathbf{r}) \simeq p_{\mathrm{F}}^{-1}\Delta_0(|\mathbf{r}|)\exp i\phi \cdot (\hat{p}_x + i\hat{p}_y)$, or equivalently

$$\Delta(r) \sim e^{i\phi} \times |\Delta| \cdot i(\nabla_x + i\nabla_y) \quad (\hat{\mathbf{p}} \equiv -i\boldsymbol{\nabla}) \tag{32}$$

[8] G. E. Volovik, JETP Letters **70**, 609 (1999).

Consider a wave packet with |momentum| $\cong$ p$_\mathrm{F}$, propagating through the origin, and write $\binom{u}{v} \equiv \exp i\mathbf{q}\cdot\mathbf{r}\binom{u'}{v'}$ $[\mathbf{q} = \hat{\boldsymbol{x}}p_\mathrm{F}]$. Then to lowest order in ∇ ("Andreev" approximation), $\hat{H}_0'$ (the effective Hamiltonian acting on $\binom{u'}{v'}$ becomes (since $p_F^{-1}(\hat{p}_x + i\hat{p}_y)\exp\, i\mathbf{q}\cdot\mathbf{r} \cong \exp\, i\mathbf{q}\cdot\mathbf{r}$)

$$\hat{H}_0'\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -iv_\mathrm{F}\partial_S & \Delta_0 \exp i\phi \\ \Delta_0 \exp -i\phi & iv_\mathrm{F}\partial_S \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} \tag{33}$$

$$(\partial_S \equiv \text{derivative along path } S)$$

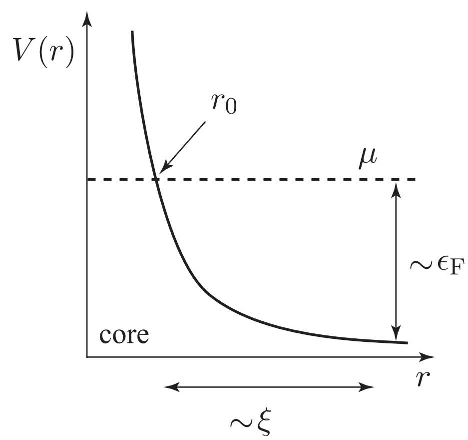
The crucial point is that since $e^{i\phi} = -1$ for $S < 0$ (L of origin) and $= +1$ for $S > 0$, this becomes the simple 1D result

$$\hat{H}_0'\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} -iv_\mathrm{F}\partial_S & \Delta_0 \,\mathrm{sgn}\, S \\ \Delta_0 \,\mathrm{sgn}\, S & iv_\mathrm{F}\partial_S \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} \tag{34}$$

This always has a zero-energy solution of the form (where the absolute phase is chosen to make $u' = v'^*$)

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \exp\frac{i\pi}{4}\cdot\begin{pmatrix} 1 \\ i \end{pmatrix}\cdot\exp\int^S ds' \,\mathrm{sgn}\, s'\Delta_0(|s'|)/v_\mathrm{F} \tag{35}$$

which is localized around origin on scale $\sim v_\mathrm{F}/\Delta_0(\infty) \sim \xi$. The usual argument is that from the continuity of the number of levels "small" perturbations to the Hamiltonian cannot remove this mode. Note the similarity in the structure of eqn. (35) to the edge modes of a topological insulator (lecture 22).

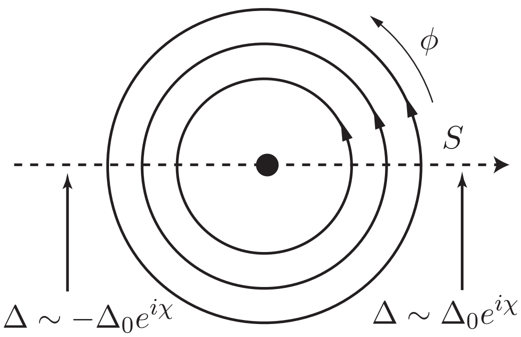


Read and Green obtain a similar result with a different model of the vortex core: $\Delta(r) \sim$ const. $= \frac{\Delta_0}{p_\mathrm{F}}(\hat{p}_x + i\hat{p}_y)$, $V(\mathbf{r})$ "$[\mu(r)]$" varying in space. Then there exists an $E = 0$ solution of the BdG equations of the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \exp i\pi/4\begin{pmatrix} 1 \\ -i \end{pmatrix}\exp -\int^r [V(r') - \mu]dr' \cdot p_\mathrm{F}/\Delta_0 \tag{36}$$

If we approximate $V(r') - \mu \simeq V'(r' - r_0) \sim \epsilon_F(r' - r_0)/\xi$, then the exponential becomes $\exp -k_0^2(r - r_0)^2$ $(k_0 \sim k_F)$. Note in this case the MF is localized within $\sim k_F$ of the core "edge," whereas in Volovik's calculations it is extended over $\sim \xi$ (and falls off as exponential, not Gaussian).

Actually, the astute reader may have noticed a prima facie problem with the above argument: why does it not apply equally to an s-wave Fermi superfluid? To be sure, in that case the $E = 0$ mode which is apparently predicted cannot be a Majorana anyon, since the superconductor in question would have to be a spin singlet and thus any fermionic quasiparticles, including $E = 0$ ones, would have spin opposite to their antiparticles and could not be Majoranas. Nevertheless, the point is worrying since



the standard result is that for a vortex in an s-wave superconductor no such $E = 0$ modes exist.

To see intuitively why there is a difference between the s-wave and p-wave cases, let us note that in view of the cylindrical symmetry of the problem there is nothing special about the x-axis, and we should therefore expect that any physical mode would be a superposition of states of the general form (35) corresponding to propagation in arbitrary directions $\mathbf{q}$; with appropriate complex amplitudes. Now in the s-wave case, where the off-diagonal terms in the Hamiltonian (33) have the form $\Delta_o e^{\pm i\varphi}$, it is clear that the spinor components (u', v') must have some nontrivial φ-dependence, and this then gives rise to an "angular" kinetic energy which cannot in general be zero. However, in the p-wave-case the factor $\nabla_x + i\nabla_j$ in the off-diagonal term $\Delta(r)$ (32) gives rise to an additional factor $q_x + iq_y$ in (33), which may[9] cancel the exp $i\varphi$, thereby allowing u' and v' to be independent of angle and the angular kinetic energy to be zero. Clearly this argument is too "hand-waving" to be entirely convincing, but a more quantitiative analysis (see Volovik, ref. cit.) leads to the same conclusion (irrespective of the relative helicity of the vortex and the Cooper pairs).

A single MF is intuitively "less than" a real (Dirac) fermion (cf. below). Where is the "rest" of it?

Theorem: MF's always come in pairs!

This is because in any given experimental geometry containing a $(p + ip)$ superfluid, either the number of vortices/antivortices is even, or the form of the OP near the container edge also sustains an MF.

But, for $2n$ vortices with $n > 1$, we do not know which MF to "pair" with which! The \$64K question is: what Berry phase does the Majorana fermion acquire when the gap Δ rotates through 2π?

[9] Provided we take the helicity of the vortex to be *opposite* to that of the Cooper pairs (i.e. $e^{i\varphi} \to e^{-i\varphi}$).

Intuitive argument: for an arbitrary "reference" phase χ we have

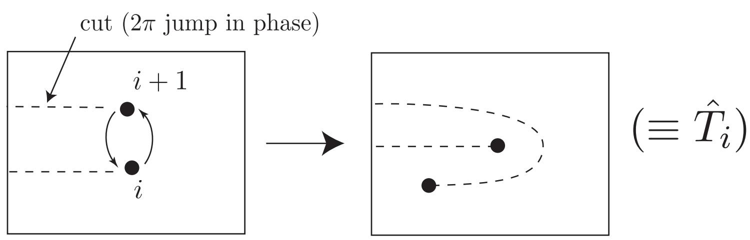
$$\hat{H}' = \begin{pmatrix} -iv_{\mathrm{F}}\partial_S & \Delta_0(S)\exp i\chi \\ \Delta_0(S)\exp -i\chi & iv_{\mathrm{F}}\partial_S \end{pmatrix} \tag{37}$$

so the generalized solution that preserves $u = v^*$ is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \exp i(\pi/4 + \chi/2) \\ \exp -i(\pi/4 + \chi/2) \end{pmatrix} \cdot \exp - \int^s ds' \Delta_0(s')/v_{\mathrm{F}} \tag{38}$$

Thus after $\chi \to \chi + 2\pi$, $\binom{u}{v} \to -\binom{u}{v}$, i.e., the Berry phase is π (just as for a regular (Bogoliubov) fermion).

Suppose now that we have a system containing $2n$ vortices. Let's number them $1, 2 \ldots 2n$ in an arbitrary way, and consider the result of interchanging vortices. Ivanov (ref. cit.) gives the following argument



The vortex i "sees" no change in the phase of the superconducting order parameter $\Delta(\mathbf{r})$, while the vortex $i+1$ sees a change of 2π. Hence the creation operators $\hat{\gamma}_i$ of the Majorana fermions transform under this exchange process (call it $\hat{T}_i$) as:

$$\hat{T}_i \begin{cases} \hat{\gamma}_i \to \hat{\gamma}_{i+1} \\ \hat{\gamma}_{i+1} \to -\hat{\gamma}_i \\ \hat{\gamma}_j \to \hat{\gamma}_j \text{ for } j \neq i,\ i+1 \end{cases} \tag{39}$$

It is interesting that the operators $\hat{T}_i$ so defined satisfy the commutation relations of the "braid group," namely

$$\begin{gathered} [\hat{T}_i, \hat{T}_j] = 0 \text{ if } |i-j| > 1 \\ \hat{T}_i\hat{T}_j\hat{T}_i = \hat{T}_j\hat{T}_i\hat{T}_j \text{ if } |i-j| = 1 \end{gathered} \tag{40}$$

Now let us consider the relation between the Majorana fermions and the real (Dirac) fermions. The latter must satisfy the standard anticommutation relations

$$\{a_i, a_j^+\} = 2\delta_{ij}, \tag{41a}$$

$$a_i^2 = a_i^{+2} = 0 \tag{41b}$$

In view of the basic ACRS $\{\psi(\mathbf{r}),\ \psi^\dagger(\mathbf{r}')\} = \delta(\mathbf{r}-\mathbf{r}')$ (etc.) and the definition (25) of the $\hat{\gamma}_i$ the latter satisfy (41a) but not (41b). However, we can make up linear combinations of $\hat{\gamma}_i$ and $\hat{\gamma}_{i+1}$, which satisfy both (41a) and (41b) and hence can represent Dirac creation and annihilation operators, as follows:

$$a_i^+ \equiv \frac{1}{\sqrt{2}}(\hat{\gamma}_i + i\hat{\gamma}_{i+1}) \qquad (42)$$
$$a_i \equiv \frac{1}{\sqrt{2}}(\hat{\gamma}_i - i\hat{\gamma}_{i+1})$$

Thus, as already noted, $2n$ Majorana fermions are equivalent to n Dirac fermions, and the relevant Hilbert space is 2^n-dimensional.

Now, what happens to the *Dirac* fermions when i and $i+1$ are interchanged? From (39) and (42) it is easy to see that they transform as follows:

$$a_i^+ \to -ia_i^+, \qquad a_i \to ia_i \qquad (43)$$

Thus, if we consider the two qubit states $|0\rangle$ and $|1\rangle$ associated with the absence and presence respectively of a Dirac fermion on vortices i and $i+1$, we have for the action of the operator $\hat{\tau}(T_i)$, which exchanges these two vortices

$$\hat{\tau}(T_i) : \begin{cases} |0\rangle \to |0\rangle \\ |1\rangle \equiv a_i^+|0\rangle \to -ia_i^+|0\rangle \equiv -i|1\rangle \end{cases}$$

It is as if "half of" the Dirac fermion had been rotated through 2π. Explicitly, the matrix representation of $\hat{\tau}(\hat{T}_i)$ is

$$\hat{\tau}(\hat{T}_i) = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \qquad (44)$$

(Alternative argument from angular momentum considerations)

At this point we notice that for $n > 1$ the association of a given pair out of the $2n$ Majorana modes to form a Dirac mode is quite arbitrary. For definiteness let us consider the case $n = 2$ and associate MF's 1 and 2 to make qubit 1 and MF's 3 and 4 to make qubit 2. Then we can represent the operator corresponding to exchange of 1 and 2, up to an irrelevant overall phase factor, as $\hat{\tau}(1 \rightleftarrows 2) = \exp i\frac{\pi}{4}\hat{\sigma}_{z1}$, and similarly the operator corresponding to exchange of 3 and 4 as $\hat{\tau}(3 \rightleftarrows 4) = \exp i\frac{\pi}{4}\hat{\sigma}_{z2}$. But what about $\hat{\tau}(2 \rightleftarrows 3)$?

Although Ivanov (ref. cit.) uses a shortcut, the most foolproof way to determine the effect of this operation is to change the basis so that the two qubits are now (1, 4) and (2, 3), so that we have in the new basis $\hat{\tau}(2 \rightleftarrows 3) = \exp i\frac{\pi}{4}\hat{\sigma}_{z2}$, and finally reverse the basis change. The result is

$$\hat{\tau}(2 \rightleftarrows 3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} \equiv \frac{1}{\sqrt{2}}(1 - i\hat{\sigma}_{x1}\sigma_{x2}) \qquad (45)$$

which is clearly entangling.

Finally, we consider the "fusion" process. Supposed we have a vortex and antivortex that recombine. They may or may not share a Dirac fermion. If they do not, they recombine to vacuum, which we denote by 1. If they do, then as they approach one another to recombine, the Dirac fermion, which for large separation of the vortices had zero energy, acquires a nonzero energy and turns into a Bogoliubov particle, which we denote ψ. Thus, denoting a vortex (antivortex) by σ, we get the "fusion rule"

$$\sigma \times \sigma = 1 + \psi \tag{46}$$

Further, two Bogoliubov quasiparticles can recombine to the vacuum, and the relevant Bogoliubov qp cannot be associated with a single vortex; thus we get two further rules

$$\begin{aligned} \psi \times \psi &= 1 \\ \psi \times \sigma &= \sigma \end{aligned} \tag{47}$$

thereby recovering the standard "Ising-anyon" results quoted in lecture 26.

To conclude this discussion, let's try to make explicit the often-quoted analogy between the $\nu = 5/2$ QHE and the $(p+ip)$ Fermi superfluid. In the former case, we saw (lecture 27) that to form a single qubit (2D Hilbert space) we needed $four$ charge $-e/4$ quasiparticles. On the other hand, in the $(p + ip)$ Fermi superfluid a single qubit is constituted by *two* half-quantum vortices, each of which (prima facie) may or may not carry a Majorana anyon. Thus, it is tempting to regard the successive creation of 4 QHE quasiparticles as corresponding to the creation of 2 vortices, each with or without an MF on it. However, this is rather misleading, because while formally the two "physical" states of the vortex pair, that is the states with **0** and 2 MF's, form a qubit, they correspond to different overall fermion number parity and hence cannot be connected by any (bulk) physical operation, in particular not by braiding (cf. eqn. (44)). It is better to consider $four$ vortices and conserve fermion number parity, thereby generating a *physical* one-qubit Hilbert space) in particular, if we consider the odd-parity sector, then for example we might as a first shot try to map the QHE state (1,2)(3,4) on to the $p + ip$ state with an extra electron on vortices 1 and 2 and none on 3 and 4; this *can* then be converted by braiding at least partly into (e.g.) (1,3) (2,4). In other words, it is a single vortex which corresponds to an $e/4$ QH quasiparticle. It would be illuminating to work out the correspondence more quantitatively, but to the best of my knowledge this has not been done, probably because of the difficulty of writing down explicit wave functions for the $(p + ip)$ superfluid as soon as nontrivial (e.g. vortex) configurations are involved.

The orthodox account: Further developments

1. *Generalization to "spinful" systems (* ^{3}He-A*,* Sr_2RuO_4*, 2-species Fermi alkali gases*):

We need to assume that to a first approximation the 2 spin species are decoupled and thus each is described by its own OP ($\Delta_\uparrow(\mathbf{r}) \neq \Delta_\downarrow(\mathbf{r})$ in general).
Consider (a)an ordinary vortex (($\Delta_\uparrow(\mathbf{r}) = \Delta_\downarrow(\mathbf{r}) \sim \exp i\varphi$) (there is lots of evidence for these in ^{3}He-A, Sr_2RuO_4). Then for each vortex we have 2 $E = 0$ modes, one for each spin species. These are still each their own antiparticles, hence "genuine" Majorana fermions, but this makes for complications in TQC. Hence, we look for (b) a "half-quantum vortex" (HQV):

$$\Delta_\downarrow(\mathbf{r}) = \text{const.}\Delta_\uparrow(\mathbf{r}) \neq \Delta_\downarrow(\mathbf{r}) \sim \exp \mathrm{i}\varphi \tag{48}$$

(Such a configuration has not to date been seen experimentally in ^{3}He-A despite searches; however, there is evidence[10] for it in Sr_2RuO_4.)
Now there is an MF associated with the $\uparrow$ species, but none for the $\downarrow$ species, so we are in business.

2. *Effect of charge (*Sr_2RuO_4*)*:

   Ivanov's argument is prima facie for a *neutral* system: it should apply to a charged system when inter-vortex distance is $\gg \xi$ (pair radius) but $\ll \lambda_L$ (London penetration depth) At distances $\gg \lambda_L$, the AB flux associated with an ordinary vortex $= \varphi_0(\equiv h/2e)$, so a quasiparticle encircling it picks up AB phase of π (this is well known). But for an HQV in a 2-species system, "induced vorticity" leads to an AB flux of $\varphi_0/2$ (i.e. $h/4e$) and so to an AB phase of $\pi/2$ for a Dirac fermion. However, since for each spin population separately there is a nonzero circulation even for $r \gg \lambda_L$, the question arises whether there is a Berry phase φ_B associated with this. In the standard BdG approach the answer is that $\varphi_B = \pi/2$, so the total phase acquired is π just as for $r \ll \lambda_L$; see however below.

I now turn to some conceptual issues concerning $p + ip$ Fermi superfluids and the Majorana fermions that may populate them.

1. *The starting ansatz for the GS MBWF*

   Consider N spinless fermions in free space (i.e., impose periodic BC's), forming Cooper pairs in a "$p + ip$" state. The standard ansatz for GS MBWF in the PC (particle-conserving) representation is, apart from normalization,

$$\Psi_N^{(0)} = \left(\sum_{\mathbf{k}} c_{\mathbf{k}}^{(0)} a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger\right)^{N/2} |\text{vac}\rangle, \qquad c_{\mathbf{k}}^{(0)} \sim |c_{\mathbf{k}}^{(0)}| \exp i\varphi_{\mathbf{k}} \tag{49}$$

[10] Jang et al., Science 331, 186 (2011)

Is this right? (Note it has $L_z = N\hbar/2$ for arbitrary small Δ) Within the standard BCS "mean-field" ansatz, we need to minimize the sum of the KE (which depends only on $\langle n_{\mathbf{k}} \rangle$) and the pairing terms, which depend on $\langle a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} a_{-\mathbf{k}'} a_{\mathbf{k}'} \rangle$. So any ansatz that gives the same values of Ψ_N for these will be, within this approximation, degenerate with Ψ_N! Consider then the ansatz

$$\Psi'_N = \left(\textstyle\sum_{\mathbf{k}>k_F} c_{\mathbf{k}} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}\right)^{N_p} \left(\textstyle\sum_{\mathbf{k}<k_F} d_{\mathbf{k}} a_{-\mathbf{k}} a_{\mathbf{k}}\right)^{N_h} \underbrace{|\text{FS}\rangle}_{\uparrow \text{ normal GS (Fermi sea)}} \tag{50}$$

where for the moment we set $N_p = N_h$, so that N is unchanged from its N-state value N_{FS}. For orientation let's provisionally go over to a BCS-like PNC representation $\Psi = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}})|\text{vac}\rangle$. Then we reproduced the "standard" values of both $\langle n_{\mathbf{k}} \rangle$ and $F_{\mathbf{k}} \equiv \langle a_{-\mathbf{k}} a_{\mathbf{k}} \rangle$ provided we choose

$$c_{\mathbf{k}} = c^{(0)}_{\mathbf{k}}, \; d_{\mathbf{k}} = \left[c^{(0)}_{\mathbf{k}}\right]^{-1} \tag{51}$$

Indeed, at first sight it looks as if all we have done is to multiply the MBWF $\Psi^{(0)}_N$ by the constant factor $\exp -i\sum_{\mathbf{k}<k_F} \varphi_{\mathbf{k}}$! However ...

Angular momentum of Ψ'_N:
Df:

$$\begin{aligned} \hat{\Omega}_p &\equiv \sum_{\mathbf{k}>k_F} c_{\mathbf{k}} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} = \sum_{\mathbf{k}>k_F} |c^{(0)}_{\mathbf{k}}| \exp i\varphi_{\mathbf{k}} \\ \hat{\Omega}_h &\equiv \sum_{\mathbf{k}<k_F} d_{\mathbf{k}} a_{-\mathbf{k}} a_{\mathbf{k}} = \sum_{\mathbf{k}<k_F} |c^{(0)}_{\mathbf{k}}|^{-1} \exp -i\varphi_{\mathbf{k}} \end{aligned} \tag{52}$$

so that

$$\Psi'_N = \hat{\Omega}_p^{N_p} \; \hat{\Omega}_h^{N_h} |\text{vac}\rangle \tag{53}$$

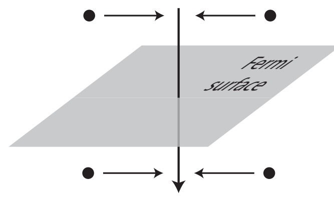
Now:

$$\left.\begin{aligned} [\hat{L}_z, \hat{\Omega}_p] &= \hbar \hat{\Omega}_p \\ [\hat{L}_z, \hat{\Omega}_h] &= -\hbar \hat{\Omega}_h \end{aligned}\right\} \text{possibly counterintuitive} \tag{54}$$

So since $|\text{FS}\rangle$ evidently has $\hat{L}_z |\text{FS}\rangle = 0$,

$$\hat{L}_z \Psi'_N = (N_p - N_h)\hbar \Psi_N = 0 \qquad \text{(in approximation } N = N_{\text{FS}}) \tag{55}$$

Caution: Ψ'_N as it stands does not reproduce $\langle V \rangle_{\text{pair}}$, because N_p and N_h are separately conserved, so that while it gives the standard values for the p-p and h-h



scattering terms, it gives zero for the p-h terms. This difficulty is easily resolved: Write $\Psi''_N = \sum_{N_p} k(N_p)\hat{\Omega}_p^{N_p}\hat{\Omega}_h^{N_p}|\text{FS}\rangle$ where $k(N_p)$ is slowly varying over N_p (range say $\sim N^{-1/2}$) and $\sum_p |k(N_p)|^2 \simeq 1$. Then the amplitude for p-h processes is proportional to $k^*(N_p)k(N_p-1) \simeq |k(N_p)|^2$, which sums to 1. Evidently, $\Psi'_N \to \Psi''_N$ does not affect the value of L_z. Thus, we have constructed an alternative GSWF that is degenerate with the standard one (within terms $\sim N^{1/2}$) but has total angular momentum zero (and hence cannot simply be a multiple of the standard one). Evidently the $64K question is, which (if either) is correct? Note that the form of the real-space many-body wavefunction has a quite different topology in the two cases, and in particular for the ansatz (50) does not have the Pfaffian form (16) at long distances.

2. *Can we do without Majorana fermions?* (indeed without "spontaneously broken $U(1)$ gauge symmetry"!)? The answer turns out to be yes. Recall the result for a translationally invariant system in simple BCS theory: (up to normalization), for even N, PC $\to \Psi_N = \left[\sum_{\mathbf{k}} c_{\mathbf{k}} a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}}\right]^{N/2} |\text{vac}\rangle$. If we select the pair of states $(\mathbf{k}, -\mathbf{k})$, this can be written ←↓

$$\Psi_N = \tilde{\Psi}_N^{(\mathbf{k})}|00\rangle_{\mathbf{k}} + c_{\mathbf{k}}\tilde{\Psi}_{N-k}^{(\mathbf{k})}|11\rangle_{\mathbf{k}} \tag{56}$$

where

$$\tilde{\Psi}_N^{(\mathbf{k})} \equiv \left(\sum_{\mathbf{k}'\neq\mathbf{k}} c_{\mathbf{k}'} a^\dagger_{\mathbf{k}'} a^\dagger_{-\mathbf{k}'}\right)^{N/2} |\text{vac}\rangle$$

or with normalization

$$\Psi_N = u^*_{\mathbf{k}} C^\dagger \tilde{\Psi}_{N-z}^{(\mathbf{k})}|00\rangle_{\mathbf{k}} + v^*_{\mathbf{k}}\tilde{\Psi}_{N-k}^{(k)}|11\rangle_{\mathbf{k}} \quad (|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1)$$

where

$$C^\dagger \equiv \mathcal{N}\left(\sum_{\mathbf{k}'\neq\mathbf{k}} c_{\mathbf{k}} a^\dagger_{\mathbf{k}'} a^\dagger_{-\mathbf{k}'}\right)$$

turns the *normalized* state $\Psi_{N-1}^{(\mathbf{k})}$ into the *normalized* state $\Psi_N^{(\mathbf{k})}$.

Now consider the $N+1$-particle states (odd total particle number). A simple ansatz for such a state is the (normalized) state

$$|N+1:\mathbf{k}\rangle = \tilde{\Psi}_N^{(\mathbf{k})}|10\rangle_{\mathbf{k}} \qquad (\text{or } \tilde{\Psi}_N^{(\mathbf{k})}|01\rangle_{\mathbf{k}}) \tag{57}$$

This is obtained from the expression (56) by the prescription

$$|N+1:\mathbf{k}\rangle = \left(u_{\mathbf{k}}a_{\mathbf{k}}^{\dagger} + v_{\mathbf{k}}a_{-\mathbf{k}}C^{\dagger}\right)\Psi_N \equiv \hat{\alpha}_{\mathbf{k}}^{\dagger}\Psi_N \tag{58}$$

Unsurprisingly, this state turns out to be an energy eigenstate with energy (relative to $E_0(N)+\mu$) of $E_{\mathbf{k}} \equiv \sqrt{(\epsilon_{\mathbf{k}}-\mu)^2+|\Delta_{\mathbf{k}}|^2}$ Note that one can form another expression of this type, namely

$$\hat{\beta}_{\mathbf{k}}^{\dagger} \equiv v^* a_{\mathbf{k}}^{\dagger} - u_{\mathbf{k}}^* a_{-\mathbf{k}}C^{\dagger} \tag{59}$$

$$\text{such that} \qquad \hat{\beta}_{\mathbf{k}}^{\dagger}\Psi_N \equiv 0$$

i.e., $\hat{\beta}_{\mathbf{k}}^{\dagger}$ is a pure annihilator. An arbitrary operator of the form $\lambda a_{\mathbf{k}}^{\dagger} + \mu a_{-\mathbf{k}}$ can be expressed as a linear combination of $\hat{\alpha}_{\mathbf{k}}^{\dagger}$ and $\hat{\beta}_{\mathbf{k}}^{\dagger}$. For each 4-D Hilbert space $(\mathbf{k}, -\mathbf{k})$ there are 2 quasiparticle creation operators and 2 pure annihilators.

Generalization to non-translationally-invariant case

Let's assume, for the moment, that the even-N groundstate is perfectly paired, i.e., that

$$\Psi_N(\equiv |N:0\rangle) = \mathcal{N}\left[\iint d\mathbf{r}d\mathbf{r}'\,K(\mathbf{r}\mathbf{r}')\psi^{\dagger}(\mathbf{r})\psi^{\dagger}(\mathbf{r}')\right] \tag{60}$$

where $K(\mathbf{r},\mathbf{r}')$ is some antisymmetric function. Then there exists a theorem[11] that we can always find an orthonormal set $\{m,\ \overline{m}\}$ (i.e. $(m,m') = (\overline{m},\overline{m}') = \delta_{mm'},\ (m,\overline{m}') = 0$) such that Ψ_N can be written

$$\Psi_n = \mathcal{N}\cdot\left(\sum_m c_m a_m^{\dagger}a_{\overline{m}}^{\dagger}\right)^{N/2}|\text{vac}\rangle \tag{61}$$

We could now proceed by analogy with the translation-invariant case by constructing the quantity $\tilde{\Psi}_N^{(m)} \equiv \left(\sum_{m'\neq m} c_m a_{m'}^{\dagger}a_{\overline{m}'}^{\dagger}\right)^{N/2}|\text{vac}\rangle$, etc. Then if we define $c_m = v_m/u_m$ as in that case, the operators $\hat{\beta}_m^{\dagger} \equiv v_m^* a_m^{\dagger} - u_m^* a_{\overline{m}}$ are pure annihilators (as of course are any linear combinations of them). However, in general, in contrast

[11] See e.g., Yang, RMP **34**, 694 (1962) lemma in Appendix A.

with the translation-invariant case, states of the form $|N_1 : m\rangle = \tilde{\Psi}_N^{(m)}|01\rangle_m$ are not energy eigenstates. The true $N+1$-particle energy eigenstates are superpositions:

$$
\begin{aligned}
|N+1 : E_n\rangle &= \sum_m q_m(E_n)|N+1 : E_m\rangle + (m \to \overline{m}) \\
&\sum_m |q_m(E_n)|^2 + (m \to \overline{m}) = 1
\end{aligned}
\tag{62}
$$

Equivalently, we can write

$$
|N+1 : E_n\rangle = \left\{ \sum_m (\tilde{u}_m a_m^\dagger + \tilde{v}_m a_{\overline{m}} C^\dagger) + (m \to \overline{m}) \right\} |\Psi_N\rangle \tag{63}
$$

$$
\equiv \int \left[u(\mathbf{r})\psi^\dagger + v(\mathbf{r})\psi(\mathbf{r})C^\dagger \right] \; |\Psi_N\rangle \qquad (\tilde{u}_m \equiv q_m u_m, \; \tilde{v}_m \equiv q_m v_m)
$$

which (apart from the PC factor $C^\dagger$) is exactly the form postulated in the BdG approach. The functions $u(\mathbf{r})$ and $v(\mathbf{r})$ are now determined by solving the BdG equations exactly as in the standard approach. But note we never had to relax particle conservation!

Nature of "Majorana Fermions"

In the standard approach, the BdG equations are equivalent to the statement that $[\hat{H}_{\text{BdG}}, \gamma_n^\dagger]$ $|\Psi_N\rangle = E_n \gamma_n^\dagger |\Psi_N\rangle$. For $E_n > 0$ the interpretation is unambiguous: $\gamma_n^\dagger|\Psi_N\rangle$ is an $N+1$-particle energy eigenstate with energy $(\mu +)E_n$ ("Dirac-Bogoliubov fermion"). But we know that if $(u, \; v)$ is a solution with $E_n > 0$, then $(v^*, \; -u^*)$ is a solution with energy eigenvalue $-E_n$. These negative energy solutions are usually interpreted in terms of the "filled Dirac sea."

However, the above equation is entirely compatible with the statement that $\gamma_n^\dagger|\Psi_N\rangle \equiv 0$! Hence, in the present PC approach, we interpret the "negative energy" $\gamma_n^\dagger$'s as pure annihilators. There must be exactly as many pure annihilators as there are DB fermion states. Suppose there exists a DB fermion with $E = 0$, and wavefunction (u, v) satisfying the BdG equations. The corresponding pure annihilator $\beta_0^\dagger$ automatically satisfies them, also with $E = 0$ (indeed any E!). Then let $\alpha_0^\dagger$ create the $E = 0$ DB fermion, and consider $\gamma_0^\dagger = e^{i\pi/4}(\alpha_0^\dagger + i\beta_0^\dagger)$. The wavefunction (u, v) corresponding to $\gamma_0^\dagger|\Psi_N\rangle$ obviously satisfies the BdG equations with $E = 0$, and moreover satisfies $u(\mathbf{r}) = v^*(\mathbf{r})$. Hence it conforms exactly to the definition of a "Majorana fermion." A second MF is generated by $e^{i\pi/4}(\alpha_0^\dagger - i\beta_0^\dagger)$.

Conclusion: In the PC representation, a "Majorana fermion" is nothing but a quantum superposition of a real "Dirac-Bogoliubov" fermion ($(N+1)$-particle energy eigenstate) and a pure annihilator. ←↓

Consider in particular the case where $\alpha_0^\dagger = \alpha_1^\dagger + i\alpha_2^\dagger$ with 1 and 2 referring to spatially distant positions. Then the two MF's will each be localized, at 1 and 2 respectively.

A crucial question is whether it is possible to rederive the Ivanov results within a properly particle-conserving formalism. At present (at least in my opinion) the jury is still out on this question...