Effects of magnetic fields and spin

Now we turn to the effects of magnetic fields; we will concentrate on the orbital coupling, since effects related to the Zeeman coupling are usually negligible. Of course, the effect will depend on the geometry: for example, in a thin film ($d \ll L_\phi$) we do not expect a (weak) field *parallel* to the film to have much effect. In general, the effect of the field will be to change the phase relation between the two time-reversed paths. A very spectacular example of this effect occurs in a cylindrical geometry with the magnetic field applied (ideally under Aharonov-Bohm conditions) along the cylinder axis (the conductivity is also measured along this axis). In this geometry, the effect of the magnetic field is to add to paths which go around the cylinder an extra phase $e \oint \mathbf{A} \cdot d\boldsymbol{l}$, which is $2\pi n(\Phi/\Phi_0)(\Phi_0 \equiv h/e)$ for a path circling the cylinder n times in a clockwise direction (anticlockwise paths contribute with a negative value of n). Thus, if there were no inelastic dephasing, the effect would be to multiply the total return amplitude coming from a pair of time-reversed paths $(n, -n)$ by the factor $\cos 2\pi n\Phi/\Phi_0$. In practice, the interference of a paths with given n will be attenuated, owing to inelastic dephasing, by a factor $\exp -2\pi nR/L_\phi(T)$. In any case, since the corresponding contribution to the return *probability* (and hence $\delta\sigma$) is proportional to $\cos^2 2\pi n\Phi/\Phi_0 \sim 1 + \cos 4\pi n\Phi/\Phi_0$, one expects to see an oscillation in the magnetoresistance with a period $\Phi_0/2$, i.e. $h/2e$. This is quite different from the effects associated with the simple AB effect, which gives a periodicity h/e. Moreover, we expect these oscillations to start to appear when $L_\phi(T)$ becomes comparable to the circumference $2\pi R$ of the cylinder and to sharpen with decreasing temperature. This is just what is seen experimentally.

In the case of a bulk geometry things are more complicated, but the general pattern of the effect of a magnetic field can be seen by the following simple argument: Although there is no unique relation between the time taken by an electron to return to the origin and the area normal to the magnetic field enclosed by its path, the "typical" distance to which it has migrated in time t is of order $\sqrt{Dt}$ and the area enclosed thus πDt; the flux inclosed is thus πBDt, so that the interference factor is

$$\cos\phi(t) \sim 2\pi\,\pi BDt/\Phi_0 \equiv \cos\left(2\pi^2 eBDt/h\right) \tag{1}$$

Thus, there is a characteristic time associated with the field:

$$\tau_H \sim (h/e)/(\pi DB) \tag{2}$$

Thus, the integral of $W(t)$ in our integral for $\delta\sigma$ (eqn. (10) of lecture 5) should be replaced by (something like)

$$\int_0^\infty dt\, W_0(t) \cos 2\pi t/\tau_H \tag{3}$$

where $W_0(t) = (Dt)^{-d/2}$ is the classical return probability. It is clear that the qualitative effect of this factor is to cut the integral off at a value $\sim \tau_H$ (and this effect survives a more rigorous treatment in which the different possible "shapes" of the orbits are properly taken into account). Equivalently, we can say that the effect is, at least qualitatively, to replace the spatial cutoff L by a "magnetic length" L_M defined to be equal to[1] $(\Phi_0/2\pi B)^{1/2}$.

Suppose now both the magnetic-field effect and the inelastic dephasing effect are present, and both the relevant lengths are $\ll$ the sample size L. Then, qualitatively speaking, if say $L_\phi \ll L_M$ than we would expect L_ϕ to dominate the behavior, which should be insensitive to L_M, and vice versa. Thus we expect that as a function of field at fixed temperature (hence fixed L_ϕ) the resistance should initially be flat, but should start to *decrease* ("negative magnetoresistance") as soon as $L_M \sim L_\phi$, and for large fields ($L_M \ll L_\phi$) should be a function only of field and independent of T. Specifically, in 2D we expect

$$\delta\sigma = -\frac{e^2}{\pi^2\hbar}\ln\left(L_\phi f(L_\phi/L_M)/l\right) \tag{4}$$

where the function $f(x)$ (which we would of course need a more quantitative calculation to determine in detail) tends to 1 for $x \ll 1$ and to x^{-1} for $x \gg 1$. Provided that the experimentally obtained $\delta\sigma$ (or equivalently δR) indeed satisfies a scaling equation of the form (4), it follows immediately that we can use the scale of the observed magnetoresistance as a direct measure of the dephasing ("phase-breaking") length $L_\phi(T)$, and by varying the temperature find its temperature-dependence. This is taken up in the next lecture.

Spin effects in weak localization

So far we have ignored the spin degree of freedom of the electron, assuming that the spin is conserved throughout the relevant trajectories. But in fact, in a realistic sample we may have both spin-orbit scattering and spin-dependent scattering by static magnetic impurities. Both of these processes are typically very weak compared to the *elastic* scattering rate τ, but need not be small compared to the phase-breaking rate τ_ϕ, so they need to be taken into account. In fact the results are quite surprising, especially in the spin-orbit case. Consider this case first.

When an electron is scattered by an atom at $\mathbf{R}_i^0$ from a state $\mathbf{k}$ into state $\mathbf{k}'$, the matrix element has the general form (in the tensor product orbital-spin space)

$$M = (V_{\mathbf{k}-\mathbf{k}'}\hat{1} + \tilde{V}_{\mathbf{k}-\mathbf{k}'}\, i\mathbf{k}\times\mathbf{k}'\cdot\hat{\boldsymbol{\sigma}})(\times\,\text{something depending on}\,\mathbf{R}_i^0) \tag{5}$$

where the second term expresses the effect of the spin-orbit interaction and generally has $\tilde{V}_{\mathbf{k}-\mathbf{k}'} \ll V_{\mathbf{k}-\mathbf{k}'}$. Because of this, the effect of spin-orbit term on the orbital motion may be neglected, so that we consider $\mathbf{k}$ and $\mathbf{k}'$ as fixed vectors; thus, intuitively, the SO interaction resembles a random magnetic field acting on the spins. However, it is crucial to appreciate that in the time-reversed trajectory the vectors $\mathbf{k}'$ and $\mathbf{k}$ are interchanged for every scattering event; thus the quantity $\mathbf{k}\times\mathbf{k}'$, i.e. the "effective magnetic field"

[1] Or some number of order 1 times this. (There seems to be no standard definition in the weak-localization context: in the theory of quantum Hall effect the standard definition is $L_M \equiv (\hbar/eB)^{1/2} \equiv (\Phi_0/2\pi B)^{1/2}$.

changes sign on the reversed trajectory; although the SO interaction is invariant under reversal of *both* the orbital and spin coordinates, it is not invariant under reversal of the orbital ones alone. As a result, if the electron spin undergoes some sequence of processes on the original trajectory, it will undergo the same sequence on the opposite trajectory *but in the reversed direction*, so that in general the spin states $|\sigma\rangle$, $|\sigma'\rangle$ at the end of the day would be different on the two trajectories (but not in general opposite). Since the interference term is proportional to the spin-space overlap $\langle\sigma|\sigma'\rangle$, one would think at first sight that the effect of SO interactions would be similar to those of inelastic collisions, i.e. they would tend to destroy the constructive interference responsible for WL, in effect adding an extra term to the dephasing rate τ_ϕ^{-1}. Moreover, supposing for the moment that they are the dominant effect in this rate (as should happen for sufficiently low T, see L. 5) one would predict that just as in the case of inelastic collisions the application of a magnetic field such that $\tau_M \lesssim \tau_\phi$ would tend to suppress the WL effect and increase σ (decrease the resistance) relative to its zero-field value.

The actual situation is much more interesting.[2] The first important consideration is that when the spin wave function of a spin-1/2 processes through an angle ϕ, it acquires a relative phase[3] of $\phi/2$, not ϕ. Hence, if for example we compute spin states which have been each rotated through π around the same axis but in opposite directions, their relative phase is now π rather than 0, and thus, if such an "effective $\pi/2$" rotation were to occur for a relevant trajectory, the effect would be to make the interference term between the original and time-reversed trajectories *negative.* Now at first sight one would think that since the amount of rotation is "random", positive interference would tend to occur at least as often as negative, so that one would not get a destructive effect on average. Formally, if we assume that we can treat the rotation angle as a simple 1D variable whose values are Gaussianly distributed, with mean square value ϕ_0^2, then we have for the average value of the interference term $\cos\phi/2$

$$\overline{\cos\phi/2} = \frac{1}{\sqrt{2\pi}\phi_0}\int_0^\infty \cos\phi/2\,\exp -\phi^2/\phi_0^2\;d\phi \sim \exp -\phi_0^2/4 \tag{6}$$

which simply decreases monotonically from 1 (the constructive interference limit) to zero as ϕ_0 increases. Thus, one would still reach the conclusion that the effect of SO interactions is, at least qualitatively, similar to that of inelastic collisions. This conclusion is in fact correct when the SO scattering is effectively 2D, as in Si MOSFET's.

What this argument misses is that the spin is actually a *three-dimensional* variable, and moreover the random "field" acting on it can, at least in the most common case when the film thickness d is $\ll l$, lie in an arbitrary direction in 3D space. As a result of this, the direction of the spin diffuses randomly on the unit sphere, and after many scatterings ($\tau \gg \tau_{\mathrm{SO}}$) it is equally likely to lie anywhere. Now it is very tempting to argue as follows: The overlap of the two arbitrary spin state should be defined, up to an overall phase factor, by $\cos\theta/2$ where θ is the angle between them. But $\theta/2$ is just the

[2]The wording of the heading of Chakravarty and Schmid §10 "Destruction of phase coherence by SO and spin-flip scattering" seems to me very misleading.

[3]This effect has been spectacularly confirmed in neutron-interferometry experiments.

angle made by one of the spins with the original axis, and the average of its cosine over the sphere is obviously zero. Hence for $\tau \gg \tau_{\rm SO}$ the average of the overlap (interference term) should be zero, as previously obtained. But this conclusion is *wrong*;[4] it neglects the interplay between the relative direction and the relative *phase* of the two spin states. In fact, the required quantity (see below) is $\frac{1}{2}{\rm Tr}\,\hat{R}^2$ where $\hat{R}$ is the total rotation matrix applied to a single spin. If we express this in terms of the standard Euler angles ϕ, θ, ψ is has the form (unitary but not Hermitian)

$$\hat{R} = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{i(\phi+\psi)/2} & i\sin\frac{\theta}{2}\, e^{-i(\phi-\psi)/2} \\ i\sin\frac{\theta}{2}\, e^{i(\phi-\psi)/2} & \cos\frac{\theta}{2}\, e^{-i(\phi+\psi)/2} \end{pmatrix} \tag{7}$$

and so it is easy to show that

$$\frac{1}{2}\,{\rm Tr}\,\hat{R}^2 = \cos^2\frac{\theta}{2}\,\cos(\phi+\psi) - \sin^2\frac{\theta}{2} \tag{8}$$

If the Euler angles θ, ϕ and ψ are assumed to be uncorrelated and random, then the first term averages to zero and $\frac{1}{2}\overline{{\rm Tr}\,R^2}$ is just the average of $\sin^2\theta/2$ over the unit sphere, i.e. $-1/2$! Thus spin-orbit scattering tends, *even in the strong-scattering limit*, to decrease the density at the origin below its classical value and hence to increase the conductivity. This behavior is sometimes called "weak antilocalization".

There is one subtlety about the above argument that needs comment: It is assumed that the spin rotations occurring on the time-reversed path occur *in reverse order* to those occurring on the original path. This means that if the total rotation on the original path is specified by Euler angles (ϕ, θ, ψ) than that on the reversed path is specified by angles $(-\psi, -\theta, -\phi)$ (not $(-\phi, -\theta, -\psi)$!). Thus, if the initial state is $|0\rangle$ we have for the final states $|\sigma\rangle_d$, $|\sigma\rangle_r$

$$|\sigma\rangle_d = \hat{R}(\phi,\theta,\psi)|0\rangle, \quad |\sigma\rangle_r = \hat{R}(-\psi,-\theta,-\phi)|0\rangle \tag{9}$$

and thus the overlap (interference) factor $\langle\sigma_r|\sigma_d\rangle$ is given by

$$I \equiv \langle\sigma_r|\sigma_d\rangle = \langle 0|\hat{R}^\dagger(-\psi,-\theta,-\phi)\hat{R}(\phi,\theta,\psi)|0\rangle = \langle 0|\hat{R}^2(\phi,\theta,\psi)|0\rangle \tag{10}$$

Taking the trace over two orthogonal possibilities for $|0\rangle$, we recover the result $I = \frac{1}{2}{\rm Tr}\,\hat{R}^2$ as stated.[5]

We now turn more briefly to the question of the effect of magnetic impurities. We represent these by an effective interaction of the form

$$\hat{H}_{\rm magn} = -\sum_i J_i \mathbf{S}_i \cdot \boldsymbol{\sigma} \tag{11}$$

[4] G. Bergmann, Solid State Communications, **42**, 815 (1982). Beware numerous typos in the derivation on p. 816!

[5] It is necessary to emphasize that we cannot simply add the effects of elastic and SO scattering: For long enough paths ($L \gtrsim L_{\rm SO}$) the *combined* effect is the SO one.

where the position $\mathbf{R}_i$ and direction of the various fixed spins (as well as, possibly, the interaction strength J_i) are random and uncorrelated. We will treat the fixed impurity spins as classical, so that the effect is, again, to provide a set of "fields", random in magnitude and direction, which act on the spin of the diffusing conduction electron. There is now a crucial difference with the SO problem, in that while the various fields again act in reverse order on the direct and time-reversed paths, they rotate in *the same* direction. Thus a *single* impurity spin would give no effect at all (i.e. whatever interference term was originally there would remain, since $|\sigma\rangle$ and $|\sigma'\rangle$, while not equal to $|0\rangle$, are identical). Any effect must come entirely from the non-Abelian structure of the relevant group ($SU(2)$), i.e. the fact that rotations do not in general commute. Thus, one expects that the effective spin-flip scattering time should be larger than the actual life time against spin-flip scattering, which is

$$\frac{1}{\tau_{\rm sf}} = \frac{2\pi}{\hbar}\frac{dn}{d\epsilon}\, n_{\rm imp} J^2 \langle S^2_{\rm imp}\rangle \tag{12}$$

but it is in general of the same order of magnitude. The qualitative effect of spin-flip scattering is similar to that of inelastic collisions. (Cf. Problem 2.1)

We have now considered the effect on weak localization of four different processes: inelastic collisions, the orbital effect of an external magnetic field, spin-orbit coupling and spin flip by static impurities. In practice all four may be simultaneously present, and the exact general formulae have to be expressed in terms of the so-called digamma function $\Psi(x)$ (in the 2D case) and look rather messy (see e.g. Bergmann, op. cit., eqn. (3.32)). However, in certain limits they simplify considerably. In the following I will for convenience neglect spin-flip scattering (as mentioned, its effect is not qualitatively very different from that of inelastic collisions, the main difference being that it is not appreciably temperature dependent and may be the dominant "phase-breaking" effect at $T \to 0$). I will also assume that all relevant lengths ($L_\phi, L_{\rm M}, L_{\rm SO}\ldots$) are large compared to the elastic mean free path l. [It should be emphasized that the condition $L_{\rm SO} \gg l$ is not necessary, and Bergmann discusses also the opposite case]. Under these conditions the basic principle is quite simple: Elastic scattering contributes positively to WL, starting at length scales $\sim$ the elastic mean free path l. Similarly, spin-orbit scattering contributes *negatively*, and in this case the smallest relevant paths are those on a length scale $L_{\rm SO}$. At the upper end, both these effects will be cut off at the length scale at which phase coherence is destroyed; this is generally the smaller of the inelastic (phase-breaking) length L_ϕ and the magnetic length $L_{\rm M}$. Below, I will generally quote the results for the 2D case, which has been most extensively investigated experimentally; the results in the 1D case are similar, with the ln's replaced by a linear dependencies (i.e. $\ln a/b \to a - b$).[6]

Consider first the case of zero magnetic field ($L_{\rm M} \to \infty$). Then apart from the elastic mean free path l, there are two characteristic lengths in the problem, namely the inelastic (phase-breaking) length L_ϕ and the spin-orbit length $L_{\rm SO}$. In the limit $L_{\rm SO} \gg L_\phi$ it is

[6] In the 2D case it clear that, whenever we have a ln with a large argument, a more exact calculation may add a constant term but will not change the argument of the logarithm.

clear that the spin-orbit effect is irrelevant, and the WL correction to the Boltzmann conductivity has the simple form

$$\Delta\sigma = -\tfrac{e^2}{\pi^2\hbar}\ln L_\phi(T)/l < 0 \quad \left[\text{or } -\tfrac{e^2}{2\pi^2\hbar}\ln \tau_\phi/\tau < 0\right] \tag{13}$$

Although we do not know the Boltzmann conductivity accurately enough to measure the absolute value of $\Delta\sigma$, we should be able to see the predicted logarithmic temperature dependence.

The opposite limit, $L_{\mathrm{SO}} \ll L_\phi(T)$ (which should be attainable at sufficiently low temperatures, since L_{SO} is nearly temperature independent while $L_\phi(T)$ diverges) is more interesting. In the range of path lengths between l and L_{SO}, the effects of the SO interaction can be neglected and the contribution to $\Delta\sigma$ is $-(e^2/\pi^2\hbar)\ln L_{\mathrm{SO}}/l \equiv -\tilde{g}_0 \ln L_{\mathrm{SO}}/l$. In the region between L_{SO} and $L_\phi(T)$, the *combined* effects of elastic and SO scattering are "antilocalizing" and contribute to $\Delta\sigma$ an amount $+\frac{1}{2}\tilde{g}_0 \ln L_\phi(T)/L_{\mathrm{SO}}$. Consequently, the complete correction to the Boltzmann conductivity has the form[7]

$$\Delta\sigma = \frac{1}{2}\frac{e^2}{\pi^2\hbar}\ln\left(L_\phi(T)l^2/L_{\mathrm{SO}}^3\right) \tag{14}$$

and is positive or negative depending on whether L_ϕ is not only $\gg L_{\mathrm{SO}}$ but $> L_{\mathrm{SO}}(L_{\mathrm{SO}}/l)^2$, a much more stringent condition. (However, the coefficient of $\ln T$ is always negative).

Now consider the effect of a magnetic field. Since the corresponding L_{M} acts as an upper cut off on the length of the paths which contribute to weak (anti-) localization, competing in that respect with $L_\phi(T)$, we can see at once that if $L_{\mathrm{M}} \gg L_\phi(T)$ $(H \to 0)$ everything should be independent of H. The opposite limit is more interesting: in this case we can simply replace $L_\phi(T)$ in the above formulae by L_{M}, and then consider the effect of varying H (i.e. L_{M}). Thus, consider

(a) $L_{\mathrm{SO}} \gg L_\phi(T)$. In this case L_{SO} is completely irrelevant, and we get:

for $L_\phi(T) \ll L_{\mathrm{M}}$, $\Delta\sigma = (-e^2/\pi^2\hbar)\ \ln L_\phi(T)/l$

for $L_\phi(T) \gg L_{\mathrm{M}}$, $\Delta\sigma = (-e^2/\pi^2\hbar)\ \ln L_{\mathrm{M}}/l$

Thus, for small H the resistivity should be approximately *independent* of H, while for large H it should decrease as const$-\ln H$. The crossover should be T-dependent, occurring when $L_{\mathrm{M}} \sim L_\phi(T)$ and thus at smaller and smaller values of H as T decreases. Of course, to obtain the exact behavior as a function of H in the crossover region one needs a more quantitative calculation.

(b) $L_{\mathrm{SO}} \ll L_\phi(T)$. This case is even more interesting. Recall that the contribution to weak localization from orbits with length scales between l and L_{SO} is positive (contribution to $\Delta\sigma$ negative), while for paths between L_{SO} and the upper cutoff the contribution is negative. Substituting L_{M} for $L_\phi(T)$ in the formulae (13) and

[7] Eqn. (10.17c) of Chakravarty and Schmid appears to be missing a factor of 1/2 in the prefactors.

(14) above we find:

$$\Delta\sigma = -\frac{e^2}{\pi^2\hbar}\ln L_{\rm M}/l \quad (L_\phi \gg)\ L_{\rm M} \gg L_{\rm SO} \tag{15}$$

$$\Delta\sigma = +\frac{1}{2}\frac{e^2}{\pi^2\hbar}\ln\left(L_{\rm M}l^2/L_{\rm SO}^3\right) \quad L_{\rm M} \ll L_{\rm SO}\ (\ll L_\phi) \tag{16}$$

and as before,

$$\Delta\sigma = -\frac{e^2}{\pi^2\hbar}\ln L_\phi/l \quad L_\phi \ll L_{\rm M} \tag{17}$$

(Note that (15) and (16) coincide when we set $L_{\rm SO} = L_{\rm M}$). Translating into expression for the change in resistance ΔR and recalling that $L_{\rm M} \propto H^{-1/2}$, we see that when plotted against $\ln H$ ΔR should initially be flat, than *increase* until (roughly) $L_{\rm M} \sim L_{\rm SO}$, and finally decrease again, eventually possibly falling below the $H = 0$ value. [Problem]

We will now examine how well the above predictions agree with experiment, restricting ourselves for the moment to the situation before $\sim$ 1995 when it started to get much more cloudy. It should be emphasized that the theoretical predictions above completely neglect the possible effects of el-el interactions, except in so far as they may contribute to τ_ϕ^{-1}; these are still somewhat controversial and will be taken up in L. 7, so it is interesting to enquire how well the "naive" theory works. A fairly complete account of the comparison of theory with experiment (as of mid-80's) can be obtained by combining Bergmann, sections 5 and 6 and Lee and Ramakrishnan section VI.

While there are some experiments on quasi-1D wires, most of experiments which have attempted to test WL theory quantitatively have been on quasi-2D systems, either Si MOSFET's, or disordered (quenched or sputtered) thin films. In both cases the "initial" (Boltzmann) $R_\square$ values are usually such that the relative effects of WL are of order $10^{-2} - 10^{-3}$. Thus it is hopeless to try to compare the *absolute* value of ΔR with theory (since we certainly do not know the Boltzmann value to this accuracy); what one studies is

(a) the qualitative features and

(b) the quantitative dependence of ΔR on temperature and magnetic field.

From the qualitative point of view, the most dramatic confirmation of the general ideas of WL theory is the experiment of Sharvin and Sharvin.[8] They measured the resistivity, in the longitudinal direction, of a thin cylindrical film of Mg deposited on a quartz filament as a function of magnetic field and thus of the flux through the cylinder. The base resistance of the film was $\sim 10\text{k}\Omega$, and the field-dependent part at low ($\lesssim 50\,$G) fields $\sim 1\,\Omega$, so 1 part in 10^4. In addition to a term in R which increased linearly with field, they observed *oscillations* which had a period $\Phi_0/2$ where Φ_0 is the single-electron

[8] JETP Letters **34**, 272 (1981).

flux quantum h/e; thus, they cannot be due to the simple AB effect, which would give a period of Φ_0.

With regard to measurements of "bulk" thin films, one should distinguish between the effects of magnetic field and temperature. In many cases, the predictions for the dependence of the resistance on magnetic field at constant T is essentially perfect; that is, once one has fitted the (constant) values of one or a few parameters (L_ϕ, $L_{\rm SO}$, possibly the spin-flip scattering length), the theoretical curves go right through the experimental points: see e.g. Bergmann section 5. Particularly impressive is the way in which the curves change from monotonic to nonmonotonic as heavy impurities like Au are added, indicating the effect of spin-orbit scattering. Another, qualitative, confirmation of the essential concepts of the theory is that the effect of a magnetic field in the plane of the film is much less than that of one normal to the plane.

With regard to the dependence on temperature the situation is less clear-cut. The first possibility is to look for the explicit temperature dependence of R in zero magnetic field. While the predicted behavior $\Delta R \sim \ln T$ is indeed seen experimentally, this is not unambiguous evidence for WL effects because it turns out that a quite independent mechanism based on interactions can give the same prediction (see LR section III.D, and lecture 7). Consequently, a more indirect method of obtaining τ_ϕ is more favorable. If one assumes that the temperature-dependence of the phase-breaking length $L_\phi(T)$ (hence of corresponding time $\tau_\phi(T)$, which is proportional to $L_\phi^2(T)$)) can indeed be obtained by fitting the magnetoresistance to the theoretical curves, and write $\tau_\phi(T) \propto T^{-p}$, then the exponent p seems to vary considerably between materials and sometimes even between different experiments on the same material; generally speaking it seems to fall in the range $1-2$. This is perhaps not too disturbing, since while the phonon contribution to τ_ϕ^{-1} is predicted to be proportional to T^3 (or T^4 at the lowest temperatures) the contribution of el-el scattering, in a dirty metal which is "3D" as regard this scattering is predicted to be proportional to $T^{3/2}$, see L. 7. In a few cases, τ_ϕ flattens off to a constant value at low T; this is what is to be expected if the inelastic length L_{sf} associated with the other possible phase-breaking mechanism, namely static magnetic impurities, and in most of those cases it seemed plausible to attribute the flattening to the presence of a small but nonzero concentration of such impurities.

Finally, as noted, there are a few experiments on 1D systems which try to probe WL. Recall that the 1D prediction is

$$\Delta\sigma = -\frac{e^2}{\pi\hbar}(\tilde{L} - l) \tag{18}$$

where of course the term in l cannot be obtained experimentally; here $\tilde{L}$ is the smaller of the actual length of the sample and the phase-breaking length. Thus one would predict that Ohm's law (that is, $R \propto L$) should hold for long wires but not for short ones, with the crossover being temperature dependent; this qualitative effect is seen, though it is not clear that there is quantitative agreement with the theory. Note by the way that the "σ" in the above formula is the 1D conductivity (i.e. total wire current/voltage gradient), so that if we infer from this the usual 3D bulk conductivity the latter contains an extra explicit factor of $1/A$. This dependence has been seen experimentally (LR, loc. cit.).

In conclusion, up to around 1995 the situation with regard to the comparison of WL theory with experiment was crudely that there was overall qualitative agreement and in some cases even spectacular quantitative agreement, and that there were essentially no experiments which seemed impossible to reconcile with the theory. As we will see later in the course, the last few years have brought a dramatic change in the situation.