Problem Sheet 3

1. "Brute-force" approach to the 1D Ising model.

   Consider an $N-$site 1D chain with Hamiltonian

$$\hat{H} = \frac{-J}{2}\sum_{i=0}^{N-1}\sigma_i\sigma_{i+1} \qquad \sigma_i = \pm 1 \qquad (J > 0) \tag{1}$$

   (a) (trivial) Write down an expression for the energy as a function of the number of "kinks" k. Is it a function of the total magnetization $M = \sum_i \sigma_i$?

   (b) Find an expression for the entropy S(M,k) in the thermodynamic limit. (Hint: Use the fact that in this limit the problem essentially reduces to the number of ways one can partition the "up" and "down" spins each into k/2 groups). Hence write down an expression for the free energy F(M,k) = E(M,k) − TS(M,k).

   (c) Show that the most probable value of k for fixed M is given by the implicit equation

$$k^2 = ((N-k)^2 - M^2)\exp - 2J/T \equiv k_m^2(M) \tag{2*}$$

   (d) Now add to the Hamiltonian a Zeeman term of the form

$$\hat{H}_2 = -\mu(\sum_i \sigma_i)\mathcal{H} \equiv -\mu M\mathcal{H} \tag{3}$$

   where $\mathcal{H}$ is the external magnetic field. By minimizing F with respect to M, find an equation similar to (2) for $\mathcal{H}$ in terms of M, k and T, and invert it to find k as $f(M, \mathcal{H}, \mathcal{T})$.

   (e) In principle, by substituting this equation into (2), we can find the equation of state, i.e. an equation for the magnetization M as a function of $\mathcal{H}$ and T. You

* Boltzmann's constant k_B is set equal to 1 to avoid confusion with k

are not required to do this, but the result should reduce to the "textbook" form.*

$$M = \frac{N sinh(\mu\mathcal{H}/k_B T)}{\sqrt{sinh^2(\mu\mathcal{H}/k_B T) + e^{-J/K_B T}}} \tag{4}$$

Check that (4) is "physically sensible" in the limits $J \ll k_B T$ and $J \gg k_B T$, and that the system is not ferromagnetic at any nonzero T.

2. (a) (very easy) By using standard Ginzburg-Landau (mean-field) theory and the definition of the superfluid density $\rho_s(T)$, show that in the limit $T \to T_c$ ρ_s has the form for both superfluid ^{4}He and superconductors (within mean-field theory)

$$\rho_s(T) = \text{const. } \rho(1 - T/T_c) \equiv \text{const. } \rho t \tag{5}$$

where ρ is the total density, and give an argument that for a translation-invariant system the constant must be of order unity.

(b) Consider liquid ^{4}He in an annular pore of (circular) cross-section $(50\text{Å})^2$ and circumference 1 cm. (This might be a primitive model of ^{4}He in Vycor). Make a rough estimate, using the result of part (a), of the value t_0 of t ($\equiv 1 - T/T_c$) at which the system loses long-range order.

(c) The same as for (b), for a film of ^{4}He of thickness 10 Å and area 1 cm^2.

(d) Are the results qualitatively affected by the replacement of the mean-field form of $\rho_s(T)$ in part (a) by what is believed to be the true behavior in the limit $T \to T_c$, namely $\rho_s(T) \sim \rho(1 - T/T_c)^\zeta$, $\zeta \approx 2/3$?

3. Consider the "Langer-Fisher" (LF) mechanism for decay of superflow, in which a (3D) vortex ring is nucleated and (may) expand to the boundaries of the system.

(a) Using the formulae of lecture 10, determine the approximate dependence of the free energy barrier for this process on the superfluid velocity v_s.

(b) For the system discussed in problem 2, part (b), at temperatures of the order of t_0, which process is more probable: the LF process, or a "bulk" phase slip

* Most standard texts on statistical mechanics derive this result by an alternative method; beware however of different definitions of J and μ.

in which a complete cross-section of the pore is turned normal?* Consider this question (i) for the lowest circulating state ($\kappa = 1$) (ii) for $v_s \sim 10\,\mathrm{m/sec}$.

(c) Under the conditions of part (b), is the system "effectively" superfluid?

Solutions due by 1 p.m. on Mon. 10 Oct.

* You may assume that so long as $v_s \ll \hbar/m\xi(T)$ where $\xi(T)$ is the GL healing length, the superflow does not contribute appreciably to the free energy of the bulk phase-slip processes.