Microscopic Properties of BCS Superconductors (cont.)

References: Tinkham, ch. 3, sections 7–9

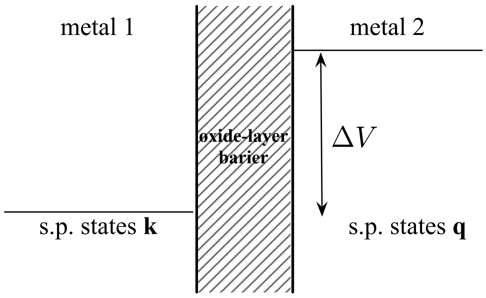
Notations: In last lecture, examined inter alia responses of system to probes that couple respectively to total **S** and total (transverse) current **J**. Both **S** and **J** have special property that they are diagonal in Bogoliubov quasiparticle operators, e.g. $\mathbf{S} = \sum_\sigma \sigma\alpha^\dagger_{\mathbf{p}\sigma}\alpha_{\mathbf{p}\sigma}$. If we consider probe which does not have this property, life becomes more complicated.

1. Tunnelling

Two metals (N or S) separated by thin oxide-layer barrier, voltage drop ΔV applied.

What current I flows from 2 to 1? Usual description of tunnelling: Bardeen-Josephson Hamiltonian:

$$\hat{H}_\mathrm{T} = \sum_{\mathbf{kq}\sigma}(T_{\mathbf{kq}\sigma}a^\dagger_{\mathbf{k}\sigma}a_{\mathbf{q}\sigma} + \mathrm{H.C.}) \quad \text{(no spin flip)} \tag{1}$$



where $\mathbf{k}$ denotes a state in 1 and $\mathbf{q}$ are in 2. Usual assumption: no special symmetries, etc., in $T_{\mathbf{kq}\sigma}$, also doesn't depend appreciably on energy. [probably OK if all relevant energies $\ll$ barrier height.]

(a) Suppose both metals are normal:

2nd order perturbation theory: (neglect any dependence of $T_{\mathbf{kq}}$ on σ).

$$\begin{aligned}
P_{\mathbf{q}\to\mathbf{k}} &= \frac{2\pi}{\hbar}|T_{\mathbf{kq}}|^2 n_{\mathbf{q}\sigma}(1-n_{\mathbf{k}\sigma})\delta(\epsilon_\mathbf{k}-\epsilon_\mathbf{q}) \\
P_{\mathbf{k}\to\mathbf{q}} &= \frac{2\pi}{\hbar}|T_{\mathbf{kq}}|^2 n_{\mathbf{k}\sigma}(1-n_{\mathbf{q}\sigma})\delta(\epsilon_\mathbf{k}-\epsilon_\mathbf{q}) \\
\Rightarrow \quad I_{2\to1} &= -\frac{2\pi e}{\hbar}\sum_{\mathbf{kq}\sigma}|T_{\mathbf{kq}}|^2(n_{\mathbf{q}\sigma}-n_{\mathbf{k}\sigma})\delta(\epsilon_\mathbf{k}-\epsilon_\mathbf{q})
\end{aligned} \tag{2}$$

Suppose average over $\hat{\mathbf{k}}, \hat{\mathbf{q}},\ \sigma$ is $\overline{|T|^2}$ and this is not a function of ϵ or ϵ', then summing over σ,

$$I = -\frac{2\pi e}{\hbar}2\overline{|T|^2}\iint d\epsilon d\epsilon'\,\rho_1(\epsilon)\rho_2(\epsilon')(n_2(\epsilon)-n_1(\epsilon'))\delta(\epsilon-\epsilon') \tag{3}$$

where $\rho_{1,2}(\epsilon)$ = single spin DOS at energy ϵ, and $n_{1,2}(\epsilon)$ = thermal (or other) occupation factor. Usually, $\rho_{1,2}(\epsilon) \sim N_{1,2}(0)$. Then

$$I = -\frac{2\pi e}{\hbar}\overline{|T|^2}2N_1(0)N_2(0)\int d\epsilon[n_2(\epsilon)-n_1(\epsilon)] \tag{4}$$

Now in thermal equilibrium, $n_{1,2} = f(\epsilon - \mu_{1,2})$, f = Fermi function. Also, to a very good approximation, $\mu_1 - \mu_2 = eV$ [note sign!]. Thus[1]

$$I = -\frac{2\pi e}{\hbar}|\overline{T}|^2 2N_1(0)N_2(0)\int_{-\infty}^{\infty} d\epsilon[f(\epsilon + eV) - f(\epsilon)] \tag{5}$$

It is clear that independently of temperature the $\int$ is simply $-eV$. Hence $G_{nn} \equiv I_{2\to 1}/V_{21}$ is given by

$$G_{nn} = \frac{2\pi e^2}{\hbar} 2N_1(0)N_2(0)|\overline{T}|^2 \tag{6}$$

The matrix element $|\overline{T}|^2$ is of course very sensitive to the details of the junction, but crucial point is that it is not expected to change (much) when one or both metals become superconductors.

(b) Now suppose one metal, for definiteness 1, is S with energy gap Δ, the metal (2) remaining N.

Assume for the moment zero temperature and let μ_2 be $> \mu_1$. The ME $T_{\mathbf{kq}}a^\dagger_{\mathbf{k}\sigma}a_{\mathbf{q}\sigma}$ must now be expressed in terms of Bogoliubov operators for metal 1:

$$T_{\mathbf{kq}}a^\dagger_{\mathbf{k}\sigma}a_{\mathbf{q}\sigma} = T_{\mathbf{k}q}(u_{\mathbf{k}}\alpha^\dagger_{\mathbf{k}\sigma} + \sigma v_{\mathbf{k}}\alpha_{-\mathbf{k},-\sigma})a_{\mathbf{q}\sigma} \tag{7}$$

The term in $v_{\mathbf{k}}$ does not contribute to $P_{\mathbf{q}\to\mathbf{k}}$ at $T = 0$, since impossible to destroy a Bogoliubov quasiparticle, so

$$P_{\mathbf{q}\to\mathbf{k}} = \frac{2\pi}{\hbar}|T_{\mathbf{kq}}|^2 u^2_{\mathbf{k}}\delta(E_{\mathbf{k}} - \epsilon_{\mathbf{q}})n_{\mathbf{q}} \tag{8}$$

The return current, $P_{\mathbf{k}\to\mathbf{q}}$, is similarly given by the Hermitian conjugate of (7) (read $v_{\mathbf{k}} = v_{-\mathbf{k}}$):

$$P_{\mathbf{k}\to\mathbf{q}} = \frac{2\pi}{\hbar}|T_{\mathbf{kq}}|^2 v^2_{\mathbf{k}}\delta(E_{\mathbf{k}} + \epsilon_{\mathbf{q}})(1 - n_{\mathbf{q}}) \tag{9}$$

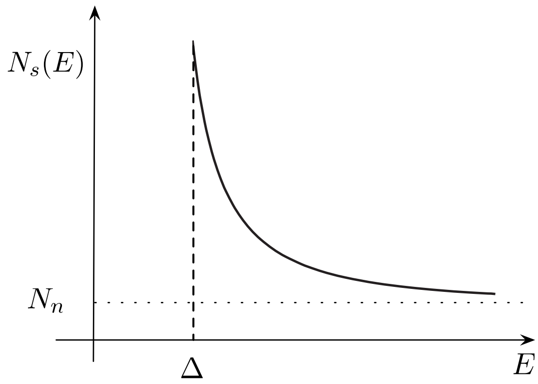
since e^- in 2 created.

If we choose eV for definiteness to be positive, then the return current is zero (at $T = 0$) and the total current $2\to 1$ is given by

$$I_{2\to 1} = -\frac{2\pi e^2}{\hbar}\sum_{\mathbf{kq}\sigma}|T_{\mathbf{kq}}|^2 u^2_{\mathbf{k}}\delta(E_{\mathbf{k}} - \epsilon_{\mathbf{q}})n_{\mathbf{q}} \tag{10}$$

where $n_{\mathbf{q}} = \theta(eV - \epsilon_{\mathbf{q}})$: note that for $V = 0$ the current vanishes as it should. Now $u^2_{\mathbf{k}} = \frac{1}{2}(1 - \epsilon_{\mathbf{k}}/E_{\mathbf{k}})$, and provided $|T_{\mathbf{kq}}|^2$ doesn't depend appreciable on $E_{\mathbf{k}}$ (usually true), the contribution of the $\epsilon_{\mathbf{k}}$-term vanishes, because states with $\pm\epsilon_{\mathbf{k}}$ have the same $E_{\mathbf{k}}$: thus we can simply set the $u^2_{\mathbf{k}} = 1/2$.

[1]Provided $eV, k_{\mathrm{B}}T \ll$ bandwith, it is clear that only states close to Fermi surface contribute.



Differentiating with respect to V, we therefore finally find:

$$G_{ns}(V) = \frac{1}{2}\frac{2\pi e^2}{\hbar}|\overline{T}|^2 N_2(0)\sum_{\mathbf{k}}\delta(E_{\mathbf{k}} - eV) \tag{11}$$

or[2]

$$G_{ns}(V)/G_{nn} = N_s(eV)/N_n(0), \quad N_s(eV) \equiv 1/2\sum_{\mathbf{k}}\delta(E_{\mathbf{k}} - eV) \tag{12}$$

so a measurement of $G_{ns}(V)$ at $T = 0$ is a direct measure of the quasiparticle DOS at excitation energy $E = eV$. Note that there are 2 values of $\epsilon_{\mathbf{k}}$ corresponding to given $E_{\mathbf{k}}$.

What does it look like? We have $N_s(E)dE = N_n d\epsilon$, so

$$N_s(E)/N_n = d\epsilon/dE = E/|\epsilon| = \frac{E}{\sqrt{E^2 - \Delta^2}} \tag{13}$$

2. Coherence factors

Consider an operator of the form

$$\sum_{\mathbf{k}\mathbf{k}'\sigma\sigma'} V_{\mathbf{k}\mathbf{k}'\sigma\sigma'}(t)a^\dagger_{\mathbf{k}\sigma}a_{\mathbf{k}'\sigma'} \equiv \hat{\Omega} \tag{14}$$

and carry out on it the Bogoliubov transformation. It will generate terms of the type(a) $\alpha\alpha$ (b) $\alpha^\dagger\alpha^\dagger$ (c) $\alpha^\dagger\alpha$ (or $\alpha\alpha^\dagger$). Terms of type (a) and (b) create or destroy 2 quasiparticles and thus correspond to a change in energy of $|E_{\mathbf{k}} + E_{\mathbf{k}'}| \geq 2\Delta(T)$. Thus, if the frequency of the perturbation V is $< 2\Delta(T)$, they will not be able to generate real transitions. On the other hand, terms of the form $\alpha^\dagger\alpha$ correspondent to scattering of Bogoliubov quasiparticles and induce transitions of frequency $E_{\mathbf{k}} - E_{\mathbf{k}'}$, which can be arbitrarily small and have either sign. However, since they involve $\alpha_{\mathbf{p}}$ they can be effective only at finite T where $\langle n_{\mathbf{p}}\rangle \neq 0$.

[2] It is convenient to put the factor of 1/2 in the df so that in the limit $\Delta \to 0$, ($E \to |\epsilon|$), we restore the normal-state result.

In transforming $\hat{\Omega}$ into Bogoliubov quasiparticle operators, we must remember that the transformation involves σ: (cf. Lecture 6).

$$a^{\dagger}_{\mathbf{k}\sigma} = u_{\mathbf{k}}\alpha^{\dagger}_{\mathbf{k}\sigma} + \sigma v_{\mathbf{k}}\alpha_{-\mathbf{k},-\sigma} \tag{15}$$

etc. Carrying out the transformation explicitly and using the ACR's:

$$\hat{\Omega} = \sum_{\mathbf{k}\mathbf{k}'\sigma\sigma'} V_{\mathbf{k}\mathbf{k}'\sigma\sigma'}(t)\Big\{u_{\mathbf{k}}u'_{\mathbf{k}}\alpha^{\dagger}_{\mathbf{k}\sigma}\alpha_{\mathbf{k}'\sigma} - v_{\mathbf{k}}v'_{\mathbf{k}}\sigma\sigma'\alpha^{\dagger}_{-\mathbf{k}',-\sigma'}\alpha_{-\mathbf{k},-\sigma} + \sigma' u_{\mathbf{k}}v_{\mathbf{k}'}\ \alpha^{\dagger}_{\mathbf{k}\sigma}\alpha^{\dagger}_{-\mathbf{k}',-\sigma'} + \sigma u_{\mathbf{k}'}v_{\mathbf{k}}\alpha_{-\mathbf{k},-\sigma}\alpha_{\mathbf{k}'\sigma}\Big\} \tag{16}$$

It is convenient to redefine the variables of summation and introduce the notation $\theta_{\sigma\sigma'} \equiv \sigma\sigma' = +1$ if $\sigma = \sigma'$, -1 if $\sigma \neq \sigma'$. We further assume (note order of indices) that

$$V_{-\mathbf{k}',-\mathbf{k},-\sigma',-\sigma} = \eta\theta_{\sigma\sigma'}V_{\mathbf{k}\mathbf{k}'\sigma\sigma'} \tag{17}$$

where $\eta = \pm 1$, i.e. V is even (type-I) or odd (type-II) under time reversal. Thus the expression for $\hat{\Omega}$ becomes

$$\hat{\Omega} = \sum_{\mathbf{k}\mathbf{k}'\sigma\sigma'} \Big\{V_{\mathbf{k}\mathbf{k}'\sigma\sigma'}(u_{\mathbf{k}}u_{\mathbf{k}'} - \eta v_{\mathbf{k}}v_{\mathbf{k}'})(\alpha^{\dagger}_{\mathbf{k}\sigma}\alpha_{\mathbf{k}'\sigma} + \eta\theta_{\sigma\sigma'}\alpha^{\dagger}_{-\mathbf{k}',-\sigma'}\alpha_{-\mathbf{k},-\sigma}) + (u_{\mathbf{k}}v_{\mathbf{k}'} + \eta v_{\mathbf{k}}u_{\mathbf{k}'})(\alpha^{\dagger}_{\mathbf{k}\sigma}\alpha^{\dagger}_{-\mathbf{k}',-\sigma'} + \eta\theta_{\sigma\sigma'}\alpha_{-\mathbf{k},-\sigma}\alpha_{\mathbf{k}'\sigma'})\Big\} \tag{18}$$

As emphasized by Tinkham, the factor $\eta\theta_{\sigma\sigma'}$ is unimportant because it relates processes which are mutually incoherent, but the factors of η in the overall coefficients are crucial. What it means is that the effective matrix for scattering of the Bogoliubov quasiparticle is multiplied, relative to that for N-state particles, by factor

$$(u_{\mathbf{k}}u_{\mathbf{k}'} - \eta v_{\mathbf{k}}v'_{\mathbf{k}}) \tag{19}$$

Thus the transition probabilities are multiplied by this factor squared:

$$(u_{\mathbf{k}}u_{\mathbf{k}'} - \eta v_{\mathbf{k}}v_{\mathbf{k}'})^2 = \frac{1}{2}(1 + \frac{\epsilon_{\mathbf{k}}\epsilon_{\mathbf{k}'}}{E_{\mathbf{k}}E_{\mathbf{k}'}} - \eta\frac{\Delta^2}{E_{\mathbf{k}}E_{\mathbf{k}'}}) \tag{20}$$

Provided that V is not strongly dependent on $\epsilon_{\mathbf{k}}$ (usually true) then when we sum over $\mathbf{k}$ and $\mathbf{k}'$ this expression multiplied by $\delta(E_{\mathbf{k}} - E_{\mathbf{k}'} - \hbar\omega)$, the $\epsilon_{\mathbf{k}}\epsilon_{\mathbf{k}'}$ terms cancel, and we are left with a factor

$$R_{\eta}(E_{\mathbf{k}}, E_{\mathbf{k}'}) = \frac{1}{2}\Big(1 - \eta\frac{\Delta^2}{E_{\mathbf{k}}E_{\mathbf{k}'}}\Big) \tag{21}$$

In the limit $\epsilon_{\mathbf{k}}, \epsilon_{\mathbf{k}'} \to 0$ ($E_{\mathbf{k}}, E_{\mathbf{k}'} \to \Delta$) this tends to 0 for $\eta = +1$ and 1 for $\eta = -1$. In the same way, the factor multiplying the two-quasiparticle creation operator term $\alpha^{\dagger}_{\mathbf{k},\sigma}\alpha^{\dagger}_{-\mathbf{k}',-\sigma'}$, namely $(u_{\mathbf{k}}v_{\mathbf{k}'} + \eta v_{\mathbf{k}}u_{\mathbf{k}'})^2$, becomes after the same summation over $\pm\epsilon_{\mathbf{k}}$,

$$\tilde{R}_{\eta} = \frac{1}{2}\Big(1 + \eta\frac{\Delta^2}{E_{\mathbf{k}}E_{\mathbf{k}'}}\Big) \tag{22}$$

which has the opposite behavior to R.

Let's now consider an expression of the general form

$$J(\omega) \equiv \sum_m Z^{-1} e^{-\beta\epsilon_m} \sum_n |\langle n| \sum_{\mathbf{kk}'\sigma\sigma'} V_{\mathbf{kk}'\sigma\sigma'} a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}'\sigma'}|0\rangle|^2 \delta\big(\omega - (\epsilon_m - \epsilon_n)\big) - (\mathbf{k}\sigma \rightleftharpoons \mathbf{k}'\sigma') \tag{23}$$

where the states m are energy eigenstates of the many-electron system with (many-electron) energies ϵ_m, and the matrix element $V_{\mathbf{kk}'\sigma\sigma'}$ is even or odd under time reversal as discussed above. Moreover, assume that the effect of averaging over the directions of $\mathbf{k}$ and $\mathbf{k}'$ doesn't introduce any special energy-dependence, so that the replacement

$$\sum_{\mathbf{kk}'\sigma\sigma'} |V_{\mathbf{kk}'\sigma\sigma'}|^2 \to (dn/d\epsilon)^2 \int d\epsilon \int d\epsilon' \, |\overline{V}|^2 \qquad (\epsilon \equiv \epsilon_{\mathbf{k}}) \tag{24}$$

is legitimate (depending on the actual structure of $V_{\mathbf{kk}'}$, the quantity $|\overline{V}|^2$ may itself involve factors of $dn/d\epsilon$). As we shall see, the expressions for the ultrasonic attenuation, nuclear spin relaxation and electromagnetic absorption are all of this type.

In the normal phase, we get, allowing for both "forward" and "backward" process,

$$\begin{aligned} J_n(\omega) &= |\overline{V}|^2 (dn/d\epsilon)^2 \iint d\epsilon d\epsilon' \, \big(f(\epsilon) - f(\epsilon')\big)\delta\big(\omega - (\epsilon' - \epsilon)\big) \\ &= |\overline{V}|^2 (dn/d\epsilon)^2 \int d\epsilon \, \big(f(\epsilon) - f(\epsilon + \hbar\omega)\big) = |\overline{V}|^2 (dn/d\epsilon)^2 \hbar\omega \end{aligned} \tag{25}$$

Now consider the S phase, and, suppose for the moment that $\omega < 2\Delta(T)$ so that only "scattering" terms contribute. The difference, now, is that all ϵ's (except for the $\int d\epsilon$'s) are replaced by the corresponding E's, and moreover the matrix element squared contains a factor $R_\eta(E, E')$.[3]

$$\begin{aligned} J_s(\omega) &= |\overline{V}|^2 (dn/d\epsilon)^2 \iint d\epsilon d\epsilon' \, \big(f(E) - f(E')\big) R_\eta(E, E') \delta\big(\omega - (E' - E)\big) \\ &= |\overline{V}|^2 (dn/d\epsilon)^2 \cdot 4 \int_\Delta^\infty dE \, \frac{E}{\epsilon}\frac{E'}{\epsilon'} \big(f(E) - f(E')\big) R_\eta(E, E') \end{aligned} \tag{26}$$

where $E' \equiv E + \hbar\omega$, $\epsilon' \equiv \epsilon'(E')$, and as above $R_\eta(E, E') \equiv \frac{1}{2}(1 - \eta\frac{\Delta^2}{EE'})$

Suppose now that we have not only $\omega < 2\Delta(T)$ but $\omega \ll \Delta(T)$. Then we can expand in ω and neglect terms proportional to ω or higher except in the difference $f(E) - f(E')$, which we approximate as $-\hbar\omega(\partial f/\partial E)$:

$$J_s(\omega) = |\overline{V}|^2 (dn/d\epsilon)^2 \int_\Delta^\infty dE \, (E^2/\epsilon^2)\Big(1 - \eta\frac{\Delta^2}{E^2}\Big)(-\partial f/\partial E) \times \hbar\omega \tag{27}$$

We see that there is a crucial difference between type-I ($\eta = +1$) and type-II ($\eta = -1$) cases. For type-I the factor $1 - \eta\Delta^2/E^2 \equiv \epsilon^2/E^2$ just cancels the E^2/ϵ^2, and we get the

[3]Note the overall factor of 4, coming from $\int_{-\infty}^{\infty} d\epsilon \to 2\int_\Delta^\infty dE \frac{E}{|\epsilon|}$

simple result ($f(\Delta) \equiv (\epsilon^{\beta\Delta} + 1)^{-1}$)

$$\frac{J_s(\omega : T)}{J_n(\omega)} = \frac{2}{e^{\beta\Delta(T)} + 1} \quad \text{(type-I)} \tag{28}$$

For the type-II case, by contrast, there is no cancellation, and we get

$$J_s(\omega) = |\overline{V}|^2 (dn/d\epsilon)^2 2 \int_\Delta^\infty dE(-\partial f/\partial E)[\frac{E^2 + \Delta^2}{E^2 - \Delta^2}] \times \hbar\omega \tag{29}$$

so that

$$\frac{J_s(\omega : T)}{J_n(\omega)} = 2 \int_\Delta^\infty dE(-\partial f/\partial E) \frac{E^2 + \Delta^2}{E^2 - \Delta^2} \quad \text{(type-II)} \tag{30}$$

This is actually logarithmically divergent in the simple isotropic-gap model: in real life the divergence is suppressed by gap anistropy, impurity scattering or, if all else fails, the finiteness of ω.

Two well known examples (needs some work, contrary to some textbooks!)

(a) US attenuation (longitudinal)[4]

The matrix element for absorption of a phonon of wave vector $\mathbf{q}$ by an electron of wave vector $\mathbf{k}$, spin σ, going off with $\mathbf{k}'$ and spin σ', is some slowly varying quantity $g_{\mathbf{kk'q}}$ [$= f(q)$ only in the simplest models] $\times$ $\delta_{\sigma\sigma'}\delta_{\mathbf{k'-k-q}}$; since $\delta_{-\mathbf{k}-(-\mathbf{k}')-\mathbf{q}} \equiv \delta_{\mathbf{k'-k+q}}$, this is a type-I operator.

We therefore only need to justify the assumption that (despite the δ-function) the angular integration leads to const $\int d\epsilon \int d\epsilon'$. The simplest way is to note that we can write formally

$$\sum_{\mathbf{kk'}} \delta_{\mathbf{k-k'-q}} |g_{\mathbf{kk'q}}|^2 = \text{const} \int d^3\mathbf{k} \int d^3\mathbf{k}'\, \delta(\mathbf{k} - \mathbf{k}' - \mathbf{q}) |g_{\mathbf{kk'q}}|^2 \tag{31}$$

$$= \text{const} \int d\epsilon \int d\epsilon' \int d\Omega_{\mathbf{k}} \int d\Omega_{\mathbf{k}'}\, \delta(k - k'\cos\theta' - q)\delta(k\sin\theta - k'\sin\theta')\delta(\phi')$$

where we took $\mathbf{q}$ to lie along the z-axis and $\mathbf{k}$ in the xz-plane. We have $k = k_F + \epsilon/v_F$, etc.: for $q \ll k_F$ (the usual situation) it is clear in the δ-function, the k_F part will dominate (since we know that ϵ is at most $\sim k_{\mathrm{B}}T/\hbar v_{\mathrm{F}}$), so to a good approximation, the angular integrals introduce no extra energy dependence: (We would need to examine the $\int$ more closely to get factors of q, etc., right).

Thus, we expect the longitudinal US attenuation to satisfy the relation

$$\alpha_S(T)/\alpha_n = \frac{2}{e^{\beta\Delta(T)} + 1} \tag{32}$$

(for a comparison with experiment, see, e.g. Kuper Fig. 12.2).

[4] Transverse US complicated by Meissner effect

(b) Nuclear spin relaxation

(treatment in some texts rather confusing!) In the original Hebel-Slichter experiments, system was allowed to come to equilibrium ($\mathbf{I} \sim \mathbf{H}$) in a finite field $H > H_c(T)$ (so sample is N). Next field was turned off (system $\to S$ state), so equilibrium value of nuclear spin $\mathbf{I}$ is zero, allowed to relax for a time t, then field switched on again and nuclear spin measured, hence obtain its decay, fitted to $I(t) = I_0 \exp(-t/T_1)$. Thus experiment measures $T_1^{-1}(T)$. In normal states, $T_1^{-1} \propto$ const T (Korringa law).

The process which relaxes I involves a nuclear spin flipping down accompanied by an electron spin flipping up, hence is proportional to the matrix element of $S^+(r) = \sum_{\mathbf{kk'}} J_{\mathbf{kk'}} \hat{\sigma}_+ a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k'}\sigma'} = \sum_{\mathbf{kk'}} J_{\mathbf{kk'}} a^\dagger_{\mathbf{k}\uparrow} a_{\mathbf{k'}\downarrow}$ where $J_{-\mathbf{k'},-\mathbf{k}} = J_{\mathbf{k},\mathbf{k'}}$. This is type-II (note role of $\theta_{\sigma\sigma'}$!), so the relation between $T_{1s}^{-1}(T)$ and the corresponding state in the normal rate at that T is

$$T_{1s}^{-1}(T)/T_{1n}^{-1}(T) = \int_0^\infty dE(-\partial f/\partial E)\frac{E^2 + \Delta^2}{E^2 - \Delta^2} \tag{33}$$

– a *rise* just below T_c (the famous Hebel-Slichter peak), followed by a drop to 0 as $T \to 0$. For comparison with experiment[5] see e.g. Rickayzen Fig. 3-4.

(c) EM absorption.

At first sight this should be similar to ultrasound absorption (since both involve the absorption of a photon (phonon) with scattering of an electron), the principal difference being that since the ME is proportional to J, a T-odd operator, it is type-II rather than type-I. Actually this analogy obscures an important difference: we have $c_s \ll v_\mathrm{F}$ but $c \gg v_\mathrm{F}$ and correspondingly, in the Sommerfeld (free-Fermi-gas) model of the N state, the US absorption is finite but the EM absorption zero! Thus to get any N-state EM absorption at all we need to take into account impurity scattering, and in general the relation between $\sigma_{1S}(\omega)$ and $\sigma_{1n}(\omega)$ is complicated, see e.g. Tinhham section 3.10.5. Simple formulas only result for $\sigma_{1n}(\omega) \sim$ const over $\sim \Delta$; this is so if $\tau_n \Delta \ll 1$ or $l \ll \xi$.

The operator corresponding to absorption of EM radiation of wave vector $\mathbf{q}$ and frequency ω is the Fourier transform of the electric current $\mathbf{j}(\mathbf{r})$, i.e.

$$\mathbf{j}_\mathbf{q} \equiv \frac{e\hbar}{m}\sum_{\mathbf{k}\sigma} \mathbf{k} a^\dagger_{\mathbf{k}+\mathbf{q}/2,\sigma} a_{\mathbf{k}-\mathbf{q}/2,\sigma} \tag{34}$$

Since the EM wave is transverse, we have to project $\mathbf{j}_\mathbf{q}$ i.e., $\mathbf{k}$, on the direction $\perp \mathbf{q} \cdot (\mathbf{j}_\mathbf{q} \cdot \mathbf{A}_\mathbf{q})$. It is clear that this perturbation is type-II. However, unlike the case of nuclear spin relaxation, the relevant frequency ω can easily be $> \Delta(T)$, so we cannot neglect 2-quasiparticle creation. At zero T, this is the only process, and

[5] In original experiment HS observed rise only by a factor ~ 2.

we get for the real part of the A.C. conductivity at $T = 0^*$

$$\sigma_{1s}/\sigma_{1n} = \int dE/|\epsilon| \int dE'/|\epsilon'|\{\tilde{R}_-(E,E')\delta(\hbar\omega - (E+E'))\} \tag{35}$$
$$\equiv \int_\Delta^\infty dE \cdot \frac{EE'}{\epsilon\epsilon'}\left(1 - \frac{\Delta^2}{EE'}\right) \quad E' \equiv E + \hbar\omega$$

It is clear that this expression vanishes for $\hbar\omega < 2\Delta(0)$: for $\hbar\omega > 2\Delta(0)$, it can be expressed as an elliptic integral (see e.g. Tinkham section 3.9.3).

At finite T, the 2-quasiparticle creation term is multiplied by a factor

$$(1-f)(1-f') - ff' = 1 - f(E) - f(E') \tag{36}$$

[note that since $f(E) < 1/2$, this is positive!]

In addition, we get a "quasiparticle scattering term" even for $\omega < 2\Delta(T)$: this has a structure qualitatively similar to that of the nuclear spin relaxation.

Further discussion of the *static* EM properties is given in Lecture 9.

*I suspect that the apparent difference of (35) from Tinkham's (3.94) is due to his use of a different sign convention for E, E'.