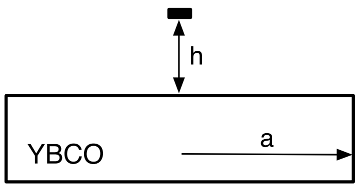
Problem Sheet 1

1. Superconducting levitation.



In a typical 'high-school laboratory' demonstration of superconducting levitation, a small magnet in a form of a disk of permalloy or some related material is placed on top of a pellet made of high temperature superconductor such as YBCO and the pellet is then submerged into liquid nitrogen. The magnet is seen to rise and float above the surface of the pellet. In the following, suppose the pellet is a flat cylinder of radius a=1cm and height 5mm, and the magnet has a mass 50mg; and that the height h at which the magnet is observed to float is 5mm.

(a) Use the above information to obtain a lower limit on the superconducting condensation (free) energy of YBCO at the boiling temperature of liquid nitrogen (77K). If we assume that the general behavior of the specific heat is similar to that of the classical superconductors, what can we infer about the condensation energy at $T = 0$? (T_c of YBCO is 92K).

(b) Find an order of magnitude for the frequency of small vertical oscillations of the magnet around its equilibrium position. (Hint: Treat the ratio h/a as 'small' and use the fact that in the limit for $h/a \to 0$ the only length relevant to the magnetostatics is h.)

(c) Estimate the order of magnitude of maximum magnetic field at the surface of the pellet, assuming the latter to be in the Meissner state. Is your estimate compatible with what we know about the magnetism of permalloy etc.?

(d) (optional, for bonus points): Can you give an argument for why the floating magnet is stable against *transverse* displacements?

2. Nonlocal electrodynamics of normal metals.

Consider the dynamics of the conduction electrons in a metal in a situation where the (local) electric field $\boldsymbol{\mathcal{E}}(\mathbf{r},t)$ varies in both space and time. Let $\delta n(\mathbf{p},\mathbf{r}:t)$ be the deviation of the semiclassical distribution function $n(\mathbf{p},\mathbf{r}:t)$ from its thermal equilibrium form $f_0(\varepsilon_{\mathbf{p}})$ where $f_0(\varepsilon)$ is the Fermi function.* The linearized Boltzmann kinetic equation may be taken for our purposes to be of the form

$$\frac{\partial}{\partial t}\,\delta n(\mathbf{r},\mathbf{p}:t) = \mathbf{v_p}\cdot\boldsymbol{\nabla}\delta n(\mathbf{r},\mathbf{p}:t) - e\left[\frac{\partial f_0}{\partial \varepsilon_{\mathbf{p}}}\right]\mathbf{v_p}\cdot\boldsymbol{\mathcal{E}}(\mathbf{r},t) - \frac{\delta n(\mathbf{r},\mathbf{p}:t)}{\tau} \qquad (1)$$

where $\mathbf{v_p} \equiv \mathbf{p}/m$ and τ is a phenomenological collision time.

(a) Show that the solution of this equation, up to additive transients, is of the form

$$\delta n(\mathbf{r},\mathbf{p}:t) = -e\,\frac{\partial f_0}{\partial \varepsilon_{\mathbf{p}}}\int_{-\infty}^{t} dt'\,\mathbf{v}\cdot\boldsymbol{\mathcal{E}}\big(\mathbf{r}-\mathbf{v_p}(t-t'),t'\big)\exp -(t-t')/\tau \qquad (2)$$

and interpret this result physically. Hence, obtain an expression for the electric current $\mathbf{j}(\mathbf{r},t)$ in terms of $\boldsymbol{\mathcal{E}}(\mathbf{r}',t')$.

(b) Consider the case of a sinusoidal varying local field,

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t) = \boldsymbol{\mathcal{E}}_0(\mathbf{r})\exp -i\omega t \qquad (3)$$

By introducing the variable $\mathbf{r}'(\mathbf{p},t-t') \equiv \mathbf{r}-\mathbf{v_p}(t-t')$, show that the Fourier transform $\mathbf{j}(\mathbf{r},\omega)$ of the current can be written in the Chambers form†

$$\mathbf{j}(\mathbf{r},\omega) = e^2\left[\frac{dn}{d\varepsilon}\right]\frac{v_{\mathrm{F}}}{4\pi}\int d\mathbf{r}'\,\frac{\mathbf{R}\big(\mathbf{R}\cdot\boldsymbol{\mathcal{E}}_0(\mathbf{r}')\big)}{R^4}\,\exp -i\omega R/v_{\mathrm{F}}\;\exp -R/l \qquad (4)$$

where $\mathbf{R}$ is a shorthand for $\mathbf{r}-\mathbf{r}'$, v_{F} is the Fermi velocity, $dn/d\varepsilon$ the density of states (of both spins) at the Fermi surface and $l \equiv v_{\mathrm{F}}\tau$ the mean free path.

(c) Show that in the limit where $\boldsymbol{\mathcal{E}}_0(\mathbf{r})$ is slowly varying over distances of the order of both t and $v_{\mathrm{F}}/\omega \equiv \lambda_\omega$, the Chambers formula reduces to the 'local' form

$$\mathbf{j}(\mathbf{r},\omega) = \sigma(\omega)\boldsymbol{\mathcal{E}}(\mathbf{r},\omega) \qquad (5)$$

* Assume that $k_{\mathrm{B}}T \ll \varepsilon_{\mathrm{F}}$ so that the usual expansion around the Fermi energy is justified.

† You may assume without proof that the correct prescription for the transformation from the integrals over t' and the direction of $\mathbf{p}$ to that over $\mathbf{r}-\mathbf{r}'$ is given by $\int dt'\int d\Omega_{\mathbf{p}} \to \int \frac{d(\mathbf{r}-\mathbf{r}')}{v_{\mathrm{F}}|\mathbf{r}-\mathbf{r}'|^2}$

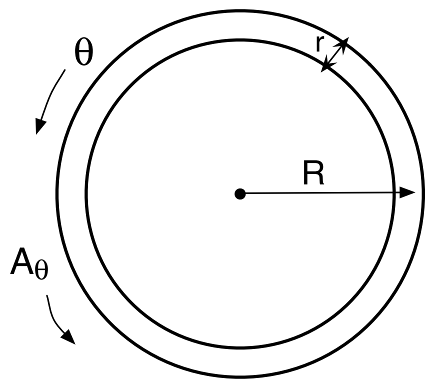
where the conductivity $\sigma(\omega)$ is given by the Drude expression:

$$\sigma(\omega) = \frac{1}{3}e^2 v_{\mathrm{F}} l \, \frac{dn}{d\varepsilon} \, \frac{1}{1+i\omega\tau} = \frac{ne^2\tau}{m} \, \frac{1}{1+i\omega\tau} \tag{6}$$

where the last expression is valid for a free-electron gas (Sommerfeld model). Use this result to rewrite the prefactor in the Chambers formula in terms of the dc conductivity $\sigma(0)$.

(d) By combining Chambers' equation with Maxwell's equations, show that the problem of penetration of a (transverse) EM field is determined, in the limit where the free-space wave length $2\pi c/\omega$ is long compared to everything else in the problem, by three characteristic lengths, namely the quantities l and λ_ω and the 'high-frequency skin depth' $\delta_0 \equiv (m/ne^2\mu_0)^{1/2}$ ($\equiv \lambda_{\mathrm{L}}(0)$ if the system becomes superconducting). By a self-consistent dimensional argument, or otherwise, find the dependence of the actual penetration depth $\delta(\omega)$ on l, λ_ω and δ_0 in the limits (i) $\lambda_\omega \gg \delta_0 \gg l$ and (ii) $l \gg \lambda_\omega \gg \delta_0$.

3. Meissner effect and flux quantization



Consider a thin metallic ring of radius R and circular cross-section πr^2. For simplicity (only) we will assume $r \ll R$ and neglect any terms of higher order than zeroth in r/R. We apply to it an external flux Φ_{ext} such that the vector potential A_{ext} is everywhere in the tangential direction and equal to $\Phi_{\mathrm{ext}}/2\pi R$ (cf. above). The effect is to replace the tangential component of momentum, P_θ, by $(P_\theta - eA_\theta)$, in both the Hamiltonian and the expression for the electric current.

(a)) By an appropriate transformation of variables, show that in classical equilibrium statistical mechanics no tangential current is induced. (Bohr-van Leeuwen theorem.)

(b) In general, a current may be induced, and will then produce an 'induced' flux Φ_{ind} which will add to the external one Φ_{ext}. Show that for $r \ll \lambda_{\mathrm{L}}(T)$ Φ_{ind} is negligible compared to Φ_{ext} provided the London equation is obeyed, and thus A may be taken constant and equal to A_{ext}.

(c) Consider the general quantum-mechanical case: write down the time-independent Schrödinger equation for the many-body system and state the boundary conditions which the wave functions must satisfy. By making an appropriate gauge transformation on the wave function, show that the free energy must be periodic in Φ_{ext} with a periodicity $\tilde{\phi}_0 = h/e$, i.e. $F(\Phi + n\tilde{\phi}_0) = F(\Phi)$.

[Note that this result, which is quite generic, is entirely compatible with (a) $F(\Phi) =$ const. (i.e. independent of Φ) and (b) $F(\Phi) =$ periodic in some submultiple of $\tilde{\phi}_0$, e.g. $h/2e$.]

(d) Consider a system of *noninteracting* QM particles in the above geometry and write down the expression for the tangential current in thermal equilibrium in the presence of Φ_{ext}. Assuming that the replacement $\sum_n f_n \to \int f(n)\, dn$ is valid provided $|f_{n+1} - f_n| \ll f_n$, find the condition for the Bohr-van Leeuwen theorem to be satisfied in this (quantum) case, assuming classical (Gibbs) statistics.‡

Now consider the case of Bose system below its transition temperature, so that a macroscopic fraction $f(T)$ of all particles must occupy the lowest energy single-particle state. Show that under these conditions the Bohr-van Leeuwen theorem is *not* satisfied and sketch the form of free energy and the current as a function of Φ. Show in particular that for $\Phi \ll \tilde{\phi}_0/2$ the current is given by the London equation, with the superfluid fraction $n_s(T)/n$ equal to $f(T)$.

Solutions to be put in 598sc homework box (2nd floor Loomis) by 9 a.m. on Mon. 15 Sept.

‡ i.e. that the probability of the occupation of a given single particle state is proportional to $\exp -\beta E_n$ where E_n is the energy of the state.